17. Vector Calculus with Applications

17.1 INTRODUCTION

In vector calculus, we deal with two types of functions: Scalar Functions (or Scalar Field) and Vector Functions (or Vector Field).

Scalar Point Function

A scalar function $F(x, y, z)$ defined over some region $R$ of space is a function which associates, to each point $P(x, y, z)$ in $R$, a scalar value $F(P) = F(x, y, z)$. And the set of all scalars $F(P)$ for all values of $P$ in $R$ is called the scalar field over $R$.

Precisely, we can say that a scalar function defines a scalar field in a region or on a space or a curve. Examples are the temperature field in a body, pressure field in the air in earth’s atmosphere.

Moreover, if the position vector of the point $P$ is $\vec{r}$, then we may also write the scalar field as $F(P) = F(\vec{r})$. This notation emphasizes the fact that the scalar value $F(\vec{r})$ is associated with the position vector $\vec{r}$ in the region $R$.

E.g. 1) The distance $F(P)$ of any point $P(x, y, z)$ from a fixed point $P'(x', y', z')$ in the space is a scalar function whose domain of definition is the whole space and is given by $F(P) = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. Also $F(P)$ defines a scalar field in space.

E.g. 2) The function $F(x, y, z) = xy^2 + yz + x^2$ for the point $(x, y, z)$ inside the unit sphere $x^2 + y^2 + z^2 = 1$ is a scalar function and also is defines a scalar field throughout the sphere.

Note: In the physical problems, the scalar function $F$ depends on time variable $t$ in addition to the point $P$ and then we write it as $F(P, t) = F(\vec{r}, t) = F(x, y, z, t)$. The example of such a time dependent scalar function is the temperature distribution throughout a block of metal heated in such a way that its temperature varies with time.

Vector Point Function

A vector functions $\vec{F}(x, y, z)$ defined over some region $R$ of space is a function which associates, to each point $P(x, y, z)$ in $R$, a vector value $\vec{F}(P) = \vec{F}(x, y, z)$ and the set of all vectors $\vec{F}(P)$ for all points $P$ in $R$ is called the vector field over $R$.

Moreover, if the position vector of the point $P$ is $\vec{r}$, then we may write the vector field as $\vec{F}(P) = \vec{F}(\vec{r})$. This notation emphasizes the fact that the vector value $\vec{F}(\vec{r})$ is associated with the position vector $\vec{r}$ in the region $R$. Also the general form (component form) of the vector function is $\vec{F}(\vec{r}) = F_1(\vec{r})\hat{i} + F_2(\vec{r})\hat{j} + F_3(\vec{r})\hat{k}$, where the components $F_1(\vec{r})$, $F_2(\vec{r})$ and $F_3(\vec{r})$ are the scalar functions.

E.g. 1) The function $\vec{F}(x, y, z) = 2xy\hat{i} + \sin x\hat{j} + 3z^2\hat{k}$ for point $P(x, y, z)$ inside an ellipsoid $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1$ is a vector function and defines a vector field throughout the ellipsoid.

E.g. 2) The force field given by $\vec{F}(x, y, z) = x\hat{i} + 2y\hat{j} + z^2\hat{k}$ is a vector field.
Note: Like the time dependent scalar field, time dependent vector field also exists. Such a field depends on time variable \( t \) in addition to the point in the region \( R \) and may be expressed as \( \vec{F}(\vec{r}, t) = F_1(\vec{r}, t)i + F_2(\vec{r}, t)j + F_3(\vec{r}, t)k \), where \( F_1, F_2 \) and \( F_3 \) are scalar functions. An example of time dependent vector field is the fluid velocity vector in the unsteady flow of water around a bridge support column, because this velocity depends on the position vector \( \vec{r} \) in the water and the time variable \( t \) and is given as \( \vec{V}(\vec{r}, t) \).

**Vector Function of Single Variable**

A vector function \( \vec{F} \) of single variable \( t \) is a function which assigns a vector value \( \vec{F}(t) \) to each scalar value \( t \) in interval \( a \leq t \leq b \). In the component form, it may be written as \( \vec{F}(t) = F_1(t)i + F_2(t)j + F_3(t)k \) where \( F_1, F_2 \) and \( F_3 \) are called components and are scalar functions of the same single variable \( t \).

For example, the functions given by \( \vec{F}(t) = t \, i + \sin(t - 2) \, j + \cos 3t \, k \) and \( \vec{G}(t) = t^2i + e^tj + \log t \, k \) are vector functions of a single variable \( t \).

**Limit of a Vector Function of Single Variable**

A vector function \( \vec{F}(t) = F_1(t)i + F_2(t)j + F_3(t)k \) of single variable \( t \) is said to have a limit \( \vec{L} = L_1i + L_2j + L_3k \) as \( t \to t_0 \), if \( \vec{F}(t) \) is defined in the neighborhood of \( t_0 \) and \( \lim_{t \to t_0} |\vec{F}(t) - \vec{L}| = 0 \) or \( \lim_{t \to t_0} |F_1(t) - L_1| = \lim_{t \to t_0} |F_2(t) - L_2| = \lim_{t \to t_0} |F_3(t) - L_3| = 0 \), then we write it as \( \lim_{t \to t_0} \vec{F}(t) = \vec{L} \).

**Continuity of a Vector Function of Single Variable**

A vector function \( \vec{F}(t) = F_1(t)i + F_2(t)j + F_3(t)k \) of a single variable \( t \) is said to be continuous at \( t = t_0 \), if it is defined in some neighborhood of \( t_0 \) and \( \lim_{t \to t_0} \vec{F}(t) = \vec{F}(t_0) \).

Moreover, \( \vec{F}(t) \) is said to be continuous at \( t = t_0 \) if and only if its three components \( F_1, F_2 \) and \( F_3 \) are continuous as \( t = t_0 \).

**DIFFERENTIAL VECTOR CALCULUS**

17.2 **DIFFERENTIATION OF VECTORES**

**Differentiability of a Vector Function of Single Variable**

A vector function \( \vec{F}(t) = F_1(t)i + F_2(t)j + F_3(t)k \) of a single variable \( t \) defined over the interval \( a \leq t \leq b \) is said to be differentiable at \( t = t_0 \) if the following limit exists.

\[
\lim_{t \to t_0} \frac{\vec{F}(t) - \vec{F}(t_0)}{t - t_0} = \vec{F}'(t_0)
\]

And \( \vec{F}'(t_0) \) is called the derivative of \( \vec{F}(t) \) at \( t = t_0 \).

Also \( \vec{F}(t) \) is said to be differentiable over the interval \( a \leq t \leq b \), if it is differentiable at each of the points of the interval. In component form, \( \vec{F}(t) \) is said to be differentiable at \( t = t_0 \) if and only if its three components are differentiable at \( t = t_0 \). In general, the derivative of \( \vec{F}(t) \) is given by...
\[ \dot{F}'(t) = \lim_{t \to t_0} \frac{\ddot{F}(t + \Delta t) - \ddot{F}(t)}{\Delta t}, \] provided the limit exists and in terms of components

\[ \ddot{F}'(t) = F_1'(t)\hat{i} + F_2'(t)\hat{j} + F_3'(t)\hat{k} \quad \text{or} \quad \frac{d\dot{F}}{dt} = \frac{dF_1}{dt}\hat{i} + \frac{dF_2}{dt}\hat{j} + \frac{dF_3}{dt}\hat{k}. \]

In the similar manner, \[ \frac{d^2 \dot{F}}{dt^2} = \frac{d}{dt} \left( \frac{d\dot{F}}{dt} \right), \frac{d^3 \dot{F}}{dt^3} = \frac{d}{dt} \left( \frac{d^2 \dot{F}}{dt^2} \right) = \frac{d^2}{dt^2} \left( \frac{d\dot{F}}{dt} \right). \]

### Rules for Differentiation of Vector Functions

If \( \vec{F}(t), \vec{G}(t) \) & \( \vec{H}(t) \) are the vector functions and \( f(t) \) is a scalar function of single variable \( t \) defined over the interval \( a \leq t \leq b \), then

1. \[ \frac{d\vec{C}}{dt} = \vec{0}, \] where \( \vec{C} \) is a constant vector.
2. \[ \frac{d}{dt} \left( C\vec{F}(t) \right) = C \frac{d\vec{F}}{dt}, \] where \( C \) is a constant.
3. \[ \frac{d}{dt} \left( \vec{F}(t) \pm \vec{G}(t) \right) = \frac{d\vec{F}}{dt} \pm \frac{d\vec{G}}{dt}. \]
4. \[ \frac{d}{dt} \left( f(t)\vec{F}(t) \right) = f(t) \frac{d\vec{F}}{dt} + \frac{df}{dt}\vec{F}(t). \]
5. \[ \frac{d}{dt} \left( \vec{F}(t) \cdot \vec{G}(t) \right) = \frac{d\vec{F}}{dt} \cdot \vec{G}(t) + \vec{F}(t) \cdot \frac{d\vec{G}}{dt}. \]
6. \[ \frac{d}{dt} \left( \vec{F}(t) \times \vec{G}(t) \right) = \frac{d\vec{F}}{dt} \times \vec{G}(t) + \vec{F}(t) \times \frac{d\vec{G}}{dt}. \]
7. \[ \frac{d}{dt} \left[ \vec{F}(t), \vec{G}(t), \vec{H}(t) \right] = \left[ \frac{d\vec{F}}{dt}, \frac{d\vec{G}}{dt}, \frac{d\vec{H}}{dt} \right] + \left[ \vec{F}(t), \frac{d\vec{G}}{dt}, \frac{d\vec{H}}{dt} \right] + \left[ \frac{d\vec{F}}{dt}, \vec{G}(t), \frac{d\vec{H}}{dt} \right]. \]
8. \[ \frac{d}{dt} \left[ \vec{F}(t) \times \left( \vec{G}(t) \times \vec{H}(t) \right) \right] = \left[ \frac{d\vec{F}}{dt} \times \left( \vec{G}(t) \times \vec{H}(t) \right) \right] + \left[ \vec{F}(t) \times \left( \frac{d\vec{G}}{dt} \times \vec{H}(t) \right) \right] + \left[ \frac{d\vec{F}}{dt} \times \left( \vec{G}(t) \times \frac{d\vec{H}}{dt} \right) \right]. \]
9. If \( \vec{F}(t) \) is differentiable function of \( t \) and \( t = t(s) \) is differentiable function then

\[ \frac{d\dot{F}}{ds} = \frac{d\dot{F}}{dt} \frac{dt}{ds}. \]

**Observations:**

(i) If \( \vec{F}(t) \) has a constant magnitude, then \( \vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \). For \( \vec{F}(t) \cdot \dot{\vec{F}}(t) = [\dot{\vec{F}}(t)]^2 = \text{constant}, \)

implying \( \vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \) or \( \vec{F} \perp \frac{d\vec{F}}{dt}. \)

(ii) If \( \vec{F}(t) \) has a constant (fixed) direction, then \( \vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}. \)

Let \( \vec{F}(t) = f(t)\vec{G}(t) \), where \( \vec{G}(t) \) is a unit vector in the direction of \( \vec{F}(t). \)

\[ \frac{d\vec{F}}{dt} = \frac{d}{dt} \left( f(t)\vec{G}(t) \right) = f(t) \frac{d\vec{G}}{dt} + \frac{df}{dt}\vec{G}(t) = \frac{df}{dt}\vec{G}(t) \quad \text{(since, \( \vec{G} \) is a constant, so \( \frac{d\vec{G}}{dt} = 0 \))} \]

and \( \vec{F} \times \frac{d\vec{F}}{dt} = f(t)\vec{G}(t) \times \frac{df}{dt}\vec{G}(t) = f(t) \frac{df}{dt} \left( \vec{G}(t) \times \vec{G}(t) \right) = \vec{0} \quad \text{(since, \( \vec{G} \times \vec{G} = \vec{0} \))} \]

**Theorem 1:** Derivative of a constant vector is a zero vector. A vector is said to be constant if both its magnitude and direction are constant (fixed).

**Proof:** Let \( \vec{F} = \vec{C} \) be a constant vector, then \( \vec{F} + \delta\vec{F} = \vec{C}. \)

On subtraction, \( \delta\vec{F} = \vec{0}. \) Which further implies that \( \frac{\delta\vec{F}}{\delta t} = \vec{0}. \)
Implying, \( \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \vec{0} \) i.e. \( \frac{d\vec{r}}{dt} = \vec{0} \).

**Theorem 2:** The necessary and sufficient condition for the vector function \( \vec{F} \) of a single variable \( t \) to have constant magnitude is \( \vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \).

**Proof:**

**Necessary condition:** Suppose \( \vec{F} \) has constant magnitude, so \( \vec{F}(t) \cdot \vec{F}(t) = [\vec{F}(t)]^2 = \text{constant} \).

\[
\Rightarrow \frac{d}{dt} (\vec{F} \cdot \vec{F}) = 0 \quad \text{i.e.} \quad \vec{F} \cdot \frac{d\vec{F}}{dt} + \frac{d\vec{F}}{dt} \cdot \vec{F} = 0
\]

\[
\Rightarrow 2\vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \quad \text{i.e.} \quad \vec{F} \cdot \frac{d\vec{F}}{dt} = 0.
\]

**Sufficient condition:** Suppose \( \vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \). \( \Rightarrow 2\vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \)

\[
\Rightarrow \vec{F} \cdot \frac{d\vec{F}}{dt} + \frac{d\vec{F}}{dt} \cdot \vec{F} = 0 \quad \Rightarrow \frac{d}{dt} (\vec{F} \cdot \vec{F}) = 0
\]

\[
\Rightarrow \vec{F} \cdot \vec{F} = \text{constant} \quad \Rightarrow |\vec{F}|^2 = \text{constant}
\]

Therefore \( \vec{F} \) has a constant magnitude.

**Theorem 3:** The necessary and sufficient condition for the vector function \( \vec{F} \) of a single variable \( t \) to have a constant direction is \( \vec{F} \times \frac{d\vec{F}}{dt} = \vec{0} \).

**Proof:** Suppose that \( \vec{f} \) is a unit vector in the direction of \( \vec{F} \) and \( \vec{F} = |\vec{F}| \), then \( \vec{f} = \frac{\vec{F}}{|\vec{F}|} \) i.e. \( \vec{F} = |\vec{F}| \vec{f} \) ... (1)

And \( \frac{d\vec{F}}{dt} = \frac{d\vec{f}}{dt} + \frac{d|\vec{F}|}{dt} \vec{f} \) ... (2)

Thus \( \vec{F} \times \frac{d\vec{F}}{dt} = \vec{F} \vec{f} \times \left( \frac{d\vec{f}}{dt} + \frac{d|\vec{F}|}{dt} \vec{f} \right) \) (using (1) and (2))

\[
= |\vec{F}|^2 \vec{f} \times \frac{d\vec{f}}{dt} + |\vec{F}| \frac{d|\vec{F}|}{dt} (\vec{f} \times \vec{f})
\]

\[
= |\vec{F}|^2 \vec{f} \times \frac{d\vec{f}}{dt} \quad \text{(since } \vec{f} \times \vec{f} = \vec{0} \text{)} \quad \cdots (3)
\]

**Necessary condition:** Suppose \( \vec{F} \) has a constant direction, then \( \vec{f} \) has a constant direction and constant magnitude. So \( \frac{d\vec{f}}{dt} = \vec{0} \). Thus from (3), \( \vec{F} \times \frac{d\vec{F}}{dt} = \vec{0} \).

**Sufficient condition:** Suppose that \( \vec{F} \times \frac{d\vec{F}}{dt} = \vec{0} \).

Then by (3), \( |\vec{F}|^2 \vec{f} \times \frac{d\vec{f}}{dt} = \vec{0} \) i.e. \( \vec{f} \times \frac{d\vec{f}}{dt} = \vec{0} \) ... (4)

Since \( \vec{f} \) has a constant magnitude, so, by theorem 2, \( \vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \) ... (5)

Form (4) and (5), \( \frac{d\vec{f}}{dt} = \vec{0} \).

Which implies \( \vec{f} \) is a constant vector i.e. \( \vec{f} \) has a constant direction. Hence \( \vec{F} \) has a constant direction.

**Example 1:** Show that if \( \vec{r} = \vec{a} \sin \omega t + \vec{b} \cos \omega t \) where \( \vec{a}, \vec{b} \) and \( \omega \) are constants, then
\[
\frac{d^2 \vec{r}}{dt^2} = -\omega^2 \vec{r} \quad \text{and} \quad \vec{r} \times \frac{d\vec{r}}{dt} = -\omega (\vec{a} \times \vec{b}).
\]

**Solution:** Given \( \vec{r} = \vec{a} \sin \omega t + \vec{b} \cos \omega t \)

Differentiating w. r. to t, \( \frac{d\vec{r}}{dt} = \vec{a} \omega \cos \omega t - \vec{b} \omega \sin \omega t \)

Again differentiating w. r. to t, \( \frac{d^2 \vec{r}}{dt^2} = -\vec{a} \omega^2 \sin \omega t - \vec{b} \omega^2 \cos \omega t \)

\[
= -\omega^2 (\vec{a} \sin \omega t + \vec{b} \cos \omega t) = -\omega^2 \vec{r}
\]

Also \( \vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a} \sin \omega t + \vec{b} \cos \omega t) \times (\vec{a} \omega \cos \omega t - \vec{b} \omega \sin \omega t) \)

\[
= (\vec{a} \times \vec{a}) \omega \sin \omega t \cos \omega t + (\vec{b} \times \vec{a}) \omega \cos^2 \omega t - (\vec{a} \times \vec{b}) \omega \sin^2 \omega t - (\vec{b} \times \vec{b}) \omega \sin \omega t \cos \omega t
\]

\[
= -(\vec{a} \times \vec{b}) \omega (\cos^2 \omega t + \sin^2 \omega t) \quad (\text{since} \ \vec{a} \times \vec{a} = \vec{b} \times \vec{b} = \vec{0})
\]

\[
= -(\vec{a} \times \vec{b}) \omega = -\omega (\vec{a} \times \vec{b}).
\]

**Example 2:** If \( \vec{a} = x^2yz \hat{i} - 2xz^3 \hat{j} + xz^2 \hat{k} \) and \( \vec{b} = 2z \hat{i} + y \hat{j} - x^2 \hat{k} \), find \( \frac{\partial^2}{\partial x \partial y} (\vec{a} \times \vec{b}) \) at (1, 0, -2).

**Solution:** Here \( \vec{a} \times \vec{b} = (x^2yz \hat{i} - 2xz^3 \hat{j} + xz^2 \hat{k}) \times (2z \hat{i} + y \hat{j} - x^2 \hat{k}) \)

\[
= x^2y^2z \hat{k} + x^4yz \hat{j} + 4xz^4 \hat{k} + 2x^2z^3 \hat{i} + 2xz^3 \hat{j} - xz^2 \hat{i}
\]

\[
= 2x^2z^3 - xyz^2 \hat{i} + (x^4yz + 2xz^3) \hat{j} + (x^2y^2z + 4xz^4) \hat{k}
\]

\[
\vdots \quad \hat{i} \times \hat{j} = \hat{k} \quad \hat{k} \times \hat{i} = \hat{j} \quad \text{and} \quad \hat{i} \times \hat{j} = \hat{k} \quad \hat{k} \times \hat{i} = \hat{j} \quad \text{etc.}
\]

\[
= (2x^3z^3 - xyz^2) \hat{i} + (x^4yz + 2xz^3) \hat{j} + (x^2y^2z + 4xz^4) \hat{k}
\]

Now \( \frac{\partial^2}{\partial x \partial y} (\vec{a} \times \vec{b}) = \frac{\partial}{\partial x} \left[ (2x^3z^3 - xyz^2) \hat{i} + (x^4yz + 2xz^3) \hat{j} + (x^2y^2z + 4xz^4) \hat{k} \right] \)

\[
= \frac{\partial}{\partial x} [(2x^3z^3 - xyz^2) \hat{i} + (x^4yz + 2xz^3) \hat{j} + (x^2y^2z + 4xz^4) \hat{k}]
\]

\[
= [ -xz^2 \hat{i} + x^4z \hat{j} + 2x^2yz \hat{k} ] = [-z^2 \hat{i} + 4x^3z \hat{j} + 4xyz \hat{k}]
\]

At the point (1, 0, -2) \( \frac{\partial^2}{\partial x \partial y} (\vec{a} \times \vec{b}) = -4 \hat{i} - 8 \hat{j} \)

**Example 3:** If \( \vec{P} = 5t^2 \hat{i} + t^3 \hat{j} - t \hat{k} \) and \( \vec{Q} = 2 \sin t \hat{i} - \cos t \hat{j} + 5t \hat{k} \), then find (a) \( \frac{d}{dt} (\vec{P} \cdot \vec{Q}) \)

(b) \( \frac{d}{dt} (\vec{P} \times \vec{Q}) \).

**Solution:** Consider \( \vec{P} = 5t^2 \hat{i} + t^3 \hat{j} - t \hat{k} \) and \( \vec{Q} = 2 \sin t \hat{i} - \cos t \hat{j} + 5t \hat{k} \)

So \( \frac{d\vec{P}}{dt} = 10t \hat{i} + 3t^2 \hat{j} - \hat{k} \) and \( \frac{d\vec{Q}}{dt} = 2 \cos t \hat{i} + \sin t \hat{j} + 5 \hat{k} \)

(a) \( \frac{d}{dt} (\vec{P} \cdot \vec{Q}) = \frac{d\vec{P}}{dt} \cdot \vec{Q} + \vec{P} \cdot \frac{d\vec{Q}}{dt} \)

\[
= (10t \hat{i} + 3t^2 \hat{j} - \hat{k}) \cdot (2 \sin t \hat{i} - \cos t \hat{j} + 5 \hat{k})
\]

\[
+ (5t^2 \hat{i} + t^3 \hat{j} - t \hat{k}) \cdot (2 \cos t \hat{i} + \sin t \hat{j} + 5 \hat{k})
\]

\[
= 20t \sin t - 3t^2 \cos t - 5t + 10t^2 \cos t + t^3 \sin t - 5t
\]

\[
= t^3 \sin t + 7t^2 \cos t + 20t \sin t - 10t
\]
b) \[ \frac{d}{dt} (\vec{p} \times \vec{q}) = \frac{d\vec{p}}{dt} \times \vec{q} + \frac{d\vec{q}}{dt} \times \vec{p} \]
\[ = (10t \hat{i} + 3t^2 \hat{j} - \hat{k}) \times (2\sin t \hat{i} - \cos t \hat{j} + 5t \hat{k}) \]
\[ + (5t^3 \hat{i} + t^3 \hat{j} - t \hat{k}) \times (2\cos t \hat{i} + \sin t \hat{j} + 5 \hat{k}) \]
\[ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10t & 3t^2 & -1 \\ 2\sin t & -\cos t & 5t \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t^3 & -t \\ 2\cos t & \sin t & 5 \end{vmatrix} \]
\[ = \{(15t^3 - \cos t) + f(-2\sin t - 50t^2) + \hat{k}(-10t \cos t - 6t^2 \sin t)\} \]
\[ + \{(5t^3 + t \sin t) + f(-2t \cos t - 25t^2) + \hat{k}(5t^2 \sin t - 2t^3 \cos t)\} \]
\[ = (20t^3 + t \sin t - \cos t) - f(2t \cos t + 2 \sin t + 75t^2) \]
\[ - \hat{k}(2t^3 \cos t + 10t \cos t + t^2 \sin t) \]

Example 4: If \[ \frac{d\vec{U}}{dt} = \vec{W} \times \vec{U} \] and \[ \frac{d\vec{V}}{dt} = \vec{W} \times \vec{V} \], then prove that \[ \frac{d}{dt} (\vec{U} \times \vec{V}) = \vec{W} \times (\vec{U} \times \vec{V}). \]

Solution: Given \[ \frac{d\vec{U}}{dt} = \vec{W} \times \vec{U} \] and \[ \frac{d\vec{V}}{dt} = \vec{W} \times \vec{V} \] \( \ldots \) (1)

Consider \[ \frac{d}{dt} (\vec{U} \times \vec{V}) = \frac{d\vec{U}}{dt} \times \vec{V} + \frac{d\vec{V}}{dt} \times \vec{U} = (\vec{W} \times \vec{U}) \times \vec{V} + \vec{W} \times (\vec{U} \times \vec{V}) \]
\[ = (\vec{W} \cdot \vec{V})\vec{U} - (\vec{U} \cdot \vec{V})\vec{W} + (\vec{U} \cdot \vec{V})\vec{W} - (\vec{U} \cdot \vec{W})\vec{V} = (\vec{W} \cdot \vec{V})\vec{U} - (\vec{U} \cdot \vec{W})\vec{V} \]
\[ = \vec{W} \times (\vec{U} \times \vec{V}) \]

(using \( \hat{a} \times \hat{b} \times \hat{c} = (\hat{a} \cdot \hat{c})\hat{b} - (\hat{b} \cdot \hat{c})\hat{a} \) and \( \hat{a} \times (\hat{b} \times \hat{c}) = (\hat{a} \cdot \hat{c})\hat{b} - (\hat{a} \cdot \hat{b})\hat{c} \))

7.3 CURVES IN SPACE

1. Tangent Vector:
Let \( \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \) be the position vector of a point P. Then for different values of the scalar parameter t, point P traces the curve in space (Fig. 7.1). For neighboring point Q with position vector \( \vec{r}(t + \delta t) \), \( \delta \vec{r} = \vec{r}(t + \delta t) - \vec{r}(t) \) implying \( \frac{\delta \vec{r}}{\delta t} = \frac{\vec{r}(t + \delta t) - \vec{r}(t)}{\delta t} \) is directed along the chord PQ.

As \( \delta t \to 0 \), \( \frac{\delta \vec{r}}{\delta t} \) becomes the tangent to the space curve at P provided there exists a non zero limit.

Thus a vector \( \frac{d\vec{r}}{dt} = \vec{r}' \) is a tangent to the space curve \( \vec{r} = \vec{r}(t) \).

Let \( P_0 \) be a fixed point on the space curve corresponding to \( t = t_0 \), and the arc length \( P_0 \vec{P} = s \), then
\[ \frac{\delta s}{\delta t} = \frac{\delta s}{|\delta \vec{r}|} \frac{|\delta \vec{r}|}{\delta t} = \frac{\text{arc} \ P \vec{Q}}{\text{chord} \ P \vec{Q}} \left| \frac{\delta \vec{r}}{\delta t} \right| \ldots \) (1)

As \( Q \to P \) along the curve QP, i.e. \( \delta t \to 0 \), then the \( \frac{\text{arc} \ P \vec{Q}}{\text{chord} \ P \vec{Q}} \to 1 \) and \( \frac{\delta s}{\delta t} = \left| \frac{d\vec{r}}{dt} \right| = |\vec{r}'(t)| \ldots \) (2)
If \( \frac{d\mathbf{r}}{dt} \) is continuous, then

\[
\mathbf{s} = \int_{t_0}^{t} \frac{d\mathbf{r}}{dt} \, dt = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
\]  

... (3)

Further, if we take \( s \) as the parameter in place of \( t \), then the magnitude of the tangent vector i.e. \( \left| \frac{d\mathbf{r}}{ds} \right| = 1 \). Thus denoting the unit tangent vector by \( \mathbf{T} \), we have

\[
\mathbf{T} = \frac{\frac{d\mathbf{r}}{ds}}{\left| \frac{d\mathbf{r}}{ds} \right|}  
\]  

... (4)

Example 5: Find the unit tangent vector at any point on the curve \( x = t^2 + 2, y = 4t - 5, z = 2t^2 - 6t \) where \( t \) is variable. Also determine the unit tangent vector at \( t = 2 \).

Solution: Let \( \mathbf{r} \) be the position vector of any point \( x, y, z \) on the given curve, then

\[
\mathbf{r} = (t^2 + 2)\mathbf{i} + (4t - 5)\mathbf{j} + (2t^2 - 6t)\mathbf{k} 
\]

The vector tangent to the curve at any point \( x, y, z \) is

\[
\frac{d\mathbf{r}}{dt} = (2t)\mathbf{i} + (4)\mathbf{j} + (4t - 6)\mathbf{k} 
\]

Now

\[
\left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(2t)^2 + (4)^2 + (4t - 6)^2} = 2\sqrt{5t^2 - 12t + 13} 
\]

Therefore unit tangent vector at \( x, y, z \)

\[
\frac{\frac{d\mathbf{r}}{dt}}{\left| \frac{d\mathbf{r}}{dt} \right|} = \frac{(2t)\mathbf{i} + (4)\mathbf{j} + (4t - 6)\mathbf{k}}{2\sqrt{5t^2 - 12t + 13}} 
\]

And the unit tangent vector at \( t = 2 \) is

\[
\left[ \frac{\frac{d\mathbf{r}}{dt}}{\left| \frac{d\mathbf{r}}{dt} \right|} \right]_{t=2} = \frac{(2t)\mathbf{i} + (4)\mathbf{j} + (4t - 6)\mathbf{k}}{2\sqrt{5(2)^2 - 12(2) + 13}}_{t=2} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}. 
\]

2. Principal Normal: Since \( \mathbf{T} \) is a unit vector, so \( \frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 0 \) i.e. either \( \frac{d\mathbf{T}}{ds} \) is perpendicular to \( \mathbf{T} \) or \( \frac{d\mathbf{T}}{ds} = 0 \), in which case \( \mathbf{T} \) is a constant vector w.r.t. the arc length \( s \) and so has a fixed direction i.e. the curve is a straight line. Now, if we denote the unit normal vector to the curve at \( P \) by \( \mathbf{N} \), then \( \frac{d\mathbf{T}}{ds} \) is in the direction of \( \mathbf{N} \) which is known as the principal normal to the curve at \( P \). The plane of \( \mathbf{T} \) and \( \mathbf{N} \) is called osculating plane of the curve at \( P \).

3. Binormal: A unit vector \( \mathbf{B} = \mathbf{T} \times \mathbf{N} \) is called the binormal at \( P \). As \( \mathbf{T} \) and \( \mathbf{N} \) both are unit vectors, so \( \mathbf{B} \) is also a unit vector normal to both \( \mathbf{T} \) and \( \mathbf{N} \) i.e. to the osculating plane of \( \mathbf{T} \) and \( \mathbf{N} \).

Thus at each point \( P \) on the curve \( C \), there are three mutually perpendicular unit vectors \( \mathbf{T}, \mathbf{N} \) and \( \mathbf{B} \), which form a moving trihedral such that

\[
\mathbf{B} = \mathbf{T} \times \mathbf{N}, \quad \mathbf{N} = \mathbf{B} \times \mathbf{T}, \quad \mathbf{T} = \mathbf{N} \times \mathbf{B} . 
\]

... (1)

This moving trihedral determines three fundamental planes at each point of the curve \( C \).

(i) The osculating plane of \( \mathbf{T} \) and \( \mathbf{N} \).

(ii) The normal plane of \( \mathbf{N} \) and \( \mathbf{B} \).

(iii) The rectifying plane of \( \mathbf{B} \) and \( \mathbf{T} \).

4. Curvature: The arc rate of turning of the tangent viz. \( \left| \frac{d\mathbf{T}}{ds} \right| \) is called the curvature of the curve and is denoted by \( \kappa \). As \( \frac{d\mathbf{T}}{ds} \) is in the direction of the principal normal \( \mathbf{N} \), therefore

\[
\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}  
\]  

... (2)
5. **Torsion:** As the binormal \( \hat{B} \) is a unit vector, so \( \frac{dB}{ds} \cdot \hat{B} = 0 \). Also \( \hat{B} \cdot \hat{T} = 0 \), therefore \( \frac{dB}{ds} \cdot \hat{T} + \hat{B} \cdot \frac{d\hat{T}}{ds} = 0 \) or \( \frac{d\hat{B}}{ds} \cdot \hat{T} + \hat{B} \cdot \frac{d\hat{T}}{ds} = 0 \). Hence, \( \frac{d\hat{B}}{ds} \) is perpendicular to both \( \hat{B} \) and \( \hat{T} \) and is, therefore, parallel to \( \hat{N} \). The arc rate of turning of the binormal viz. \( \left| \frac{d\hat{B}}{ds} \right| \) is called torsion of the curve and is denoted by \( \tau \). So, we can write it as

\[
\frac{d\hat{B}}{ds} = -\tau \hat{N} \text{ (the negative sign indicates that for } \tau > 0, \frac{d\hat{B}}{ds} \text{ has a direction of } -\hat{N}) \quad \ldots (3)
\]

Further, we know that \( \hat{N} = \hat{B} \times \hat{T} \), which on differentiation gives,

\[
\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} = (\tau \hat{T} - \kappa \hat{N})
\]

\[
\frac{d\hat{N}}{ds} = \tau \hat{B} - \kappa \hat{T} \quad \text{(using (1))} \quad \ldots (4)
\]

The relations in (1), (2) and (3) constitutes the well known **Frenet Formulas** for the curve C.

**Observations:**
(i) \( \rho = \frac{1}{\kappa} \) is called the radius of curvature.
(ii) \( \sigma = \frac{1}{\tau} \) is called the radius of torsion.
(iii) \( \tau = 0 \) for a plane curve.

**Example 6:** Find \( \hat{N}(t) \) and \( \hat{N}(1) \) for the curve represented by \( \vec{r}(t) = 3t \hat{i} + 2t^2 \hat{j} \).

**Solution:** For given \( \vec{r}(t) \), we have \( \vec{r}'(t) = 3 \hat{i} + 4t \hat{j} \) and \( |\vec{r}'(t)| = \sqrt{9 + 16t^2} \)
Which implies that the unit tangent vector \( \hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{3 \hat{i} + 4t \hat{j}}{\sqrt{9 + 16t^2}} \) \( \ldots (1) \)

Differentiating \( \hat{T}(t) \) w. r. to \( t \), \( \hat{T}'(t) = \frac{1}{\sqrt{9 + 16t^2}} (4 \hat{j}) - \frac{16t}{(9 + 16t^2)^{3/2}} (3 \hat{i} + 4t \hat{j}) = \frac{12(-4t^2 + 3)}{(9 + 16t^2)^{3/2}} \) \( \ldots (2) \)

And \( |\hat{T}'(t)| = 12 \sqrt{\frac{9 + 16t^2}{(9 + 16t^2)^3}} = \frac{12}{9 + 16t^2} \) \( \ldots (3) \)

Therefore, the principal unit normal vector is \( \hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} = \frac{-4t \hat{i} + 3 \hat{j}}{\sqrt{9 + 16t^2}} \) \( \ldots (4) \)

But at \( t = 1 \), the principal unit normal vector is \( \hat{N}(1) = \frac{1}{5} (-4\hat{i} + 3\hat{j}) \).

**Example 7:** Find the angle between the tangents to the curve \( \vec{r} = t^2 \hat{i} + 2t \hat{j} - t^3 \hat{k} \) at the point \( t = \pm 1 \).

**Solution:** Differentiating the given curve w. r. to \( t \), we get

\[
\frac{d\vec{r}}{dt} = 2t \hat{i} + 2 \hat{j} - 3t^2 \hat{k} \quad \text{which is the tangent vector to the curve at any point } t.
\]

Let \( \vec{T}_1 \) & \( \vec{T}_2 \) are the tangent vectors to the curve at \( t = 1 \) and \( t = -1 \) respectively, then

\( \vec{T}_1 = 2 \hat{i} + 2 \hat{j} - 3 \hat{k} \) and \( \vec{T}_2 = -2 \hat{i} + 2 \hat{j} - 3 \hat{k} \)

Let \( \theta \) be the angle between the tangents \( \vec{T}_1 \) & \( \vec{T}_2 \), then

\[
\cos \theta = \frac{\vec{T}_1 \cdot \vec{T}_2}{|\vec{T}_1| |\vec{T}_2|} = \frac{(2 \hat{i} + 2 \hat{j} - 3 \hat{k}) \cdot (-2 \hat{i} + 2 \hat{j} - 3 \hat{k})}{|2 \hat{i} + 2 \hat{j} - 3 \hat{k}| |2 \hat{i} + 2 \hat{j} - 3 \hat{k}|} = \frac{-4 + 4 + 9}{\sqrt{17} \sqrt{17}} = \frac{9}{17}
\]
Example 8: Find the curvature and torsion of the curve \( x = a \cos t, \ y = a \sin t, \ z = bt \).
(This curve is drawn on a circular cylinder cutting its generators at a constant angle and is known as a circular helix)

Solution: Equation of the given curve in vector form is
\[
\vec{r} = a \cos t \ \hat{i} + a \sin t \ \hat{j} + bt \ \hat{k}
\]
Differentiating w. r. to \( t \),
\[
\frac{d\vec{r}}{dt} = -a \sin t \ \hat{i} + a \cos t \ \hat{j} + b \ \hat{k}
\]
Now, the arc length of the curve from \( P_0 \) (\( t = 0 \)) to any point \( P \) (\( t \)) is given by
\[
s = \int_0^t \left| \frac{d\vec{r}}{dt} \right| \ dt = \sqrt{(a^2 + b^2)} \ t
\]
\[
\therefore \frac{ds}{dt} = \sqrt{(a^2 + b^2)}
\]
Now, the unit tangent vector,
\[
\hat{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{-a \sin t \ \hat{i} + a \cos t \ \hat{j} + b \ \hat{k}}{\sqrt{(a^2+b^2)}}
\]
So
\[
\frac{d\hat{T}}{ds} = \frac{d\hat{T}/dt}{ds/dt} = \frac{-a \cos t - a \sin t \ \hat{j}}{a^2+b^2}
\]
\[
\therefore \kappa = \left| \frac{d\hat{T}}{ds} \right| = \frac{a}{a^2+b^2} \text{ is the curvature of the given curve.}
\]
Also, the unit normal vector is \( \vec{N} = -(\cos t \ \hat{i} + \sin t \ \hat{j}) \) and
\[
\vec{B} = \hat{T} \times \vec{N} = \frac{(b \sin t \ \hat{i} - b \cos t \ \hat{j} + a \ \hat{k})}{\sqrt{a^2+b^2}}
\]
So
\[
\frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt} = \frac{b (\cos t \ \hat{i} + \sin t \ \hat{j})}{\sqrt{a^2+b^2}} = -\tau \vec{N} = \tau (\cos t \ \hat{i} + \sin t \ \hat{j})
\]
Hence \( \tau = \frac{b}{a^2+b^2} \).

Example 9: A circular helix is given by the equation \( \vec{r} = 2 \cos t \ \hat{i} + 2 \sin t \ \hat{j} + \hat{k} \). Find the curvature and torsion of the curve at any point and show that they are constant.

Solution: Equation of the given curve in vector form is
\[
\vec{r} = 2 \cos t \ \hat{i} + 2 \sin t \ \hat{j} + \hat{k}
\]
Differentiating w. r. to \( t \),
\[
\frac{d\vec{r}}{dt} = -2 \sin t \ \hat{i} + 2 \cos t \ \hat{j} + 0 \ \hat{k}
\]
Now, the arc length of the curve from \( P_0 \) (\( t = 0 \)) to any point \( P \) (\( t \)) is given by
\[
s = \int_0^t \left| \frac{d\vec{r}}{dt} \right| \ dt = 2 \ t \quad \text{implying} \quad \frac{ds}{dt} = 2
\]
Now, the unit tangent vector,
\[
\hat{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{-2 \sin t \ \hat{i} + 2 \cos t \ \hat{j} + 0 \ \hat{k}}{2}
\]
So
\[
\frac{d\hat{T}}{ds} = \frac{d\hat{T}/dt}{ds/dt} = \frac{-\cos t \ \hat{i} - \sin t \ \hat{j}}{2}
\]
\[
\therefore \kappa = \left| \frac{d\hat{T}}{ds} \right| = \frac{1}{2} \text{ is the curvature of the given curve and is a constant.}
\]
Also, the unit normal vector is \( \vec{N} = -(\cos t \ \hat{i} + \sin t \ \hat{j}) \) and
\[
\vec{B} = \hat{T} \times \vec{N} = (-\sin t \ \hat{i} + \cos t \ \hat{j}) \times (-\cos t \ \hat{i} - \sin t \ \hat{j}) = \hat{k}
\]
So
\[
\frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt} = 0 = -\tau \vec{N} = \tau (\cos t \ \hat{i} + \sin t \ \hat{j})
\]
Hence $\tau = 0$ is the torsion of the given curve and is constant.

**Example 10:** Show that for the curve $\vec{r} = a(3t - t^3)\hat{i} + 3at^2 \hat{j} + a(3t + t^3)\hat{k}$, the curvature equals torsion.

**Solution:** Given curve is $\vec{r} = a(3t - t^3)\hat{i} + 3at^2 \hat{j} + a(3t + t^3)\hat{k}$

Differentiating w. r. t, $\frac{d\vec{r}}{dt} = a(3 - 3t^2)\hat{i} + 6at \hat{j} + a(3 + 3t^2)\hat{k}$

Now, the arc length of the curve $P_0$ (t = 0) to any point $P$ (t) is given by

$$s = \int_0^t \left| \frac{d\vec{r}}{dt} \right| dt = \int_0^t \sqrt{(a(3 - 3t^2))^2 + (6at)^2 + (a(3 + 3t^2))^2} dt$$

$$= 3a\sqrt{2} \int_0^t (t^2 + 1) dt = 3a\sqrt{2} \left( \frac{t^3}{3} + t \right)$$

Hence curvature equals torsion for the given curve.
4. Find the unit tangent vector at any point on the curve \( x = t^2 + 2, y = 4t - 5, z = 2t^2 - 6t \), where \( t \) is any variable. Also determine the unit tangent vector at the point \( t = 2 \).

5. If \( \vec{r} = a \cos t \hat{i} + a \sin t \hat{j} + at \tan \alpha \hat{k} \), find the value of \( (a) \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \) \( (b) \frac{d\vec{r}}{dt} \frac{d^2\vec{r}}{dt^2} \frac{d^3\vec{r}}{dt^3} \).

   Also find the unit tangent vector at any point \( t \) on the curve.

6. Find the equation of the osculating plane and binormal to the curve

   \( (a) \vec{r} = a \cos \theta \vec{i} + \vec{b} \sin \theta \vec{j} + \vec{a} t \tan \alpha \vec{k} \).

7. Find the curvature of the \( (a) \) ellipse \( \vec{r} = a \cos \theta \vec{i} + \vec{b} \sin \theta \vec{j} \) \( (b) \) parabola \( \vec{r} = 2t \hat{i} + t^2 \hat{j} \) at point \( t = 1 \).

17.4 VELOCITY AND ACCELERATION

1. Velocity: Let the position of particle P at a time \( t \) on the curve \( C \) is \( \vec{r}(t) \) and it comes to point Q at time \( t + \delta t \) having position \( \vec{r}(t + \delta t) \), then \( \delta \vec{r} = \vec{r}(t + \delta t) - \vec{r}(t) \) \( i.e. \frac{\delta \vec{r}}{\delta t} \) is directed along PQ.

   As \( Q \rightarrow P \) along \( C \), the line PQ becomes tangent at \( P \) to the curve \( C \).

   So \( \vec{v}(t) = \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} \) is the tangent vector to \( C \) at point \( P \) which is the velocity vector \( \vec{v}(t) \) of the motion and its magnitude gives the speed \( v = \frac{ds}{dt} \), where \( s \) is the arc length of \( P \) from a fixed point \( P_0 (s=0) \) on \( C \).

2. Acceleration: Acceleration vector \( \vec{a}(t) \) of a particle is the derivative of the velocity vector \( \vec{v}(t) \), and it is given by \( \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \). It is an interesting fact that the magnitude of acceleration is not always the rate of change of \( v = |\vec{v}| \), as \( \vec{a}(t) \) is not always tangential to the curve \( C \). There are two components of acceleration, which are given as: (i) Tangential Acceleration (ii) Normal Acceleration

   Observation: The acceleration is the time rate of change of \( |\vec{v}(t)| = \frac{ds}{dt} \), if and only if the normal acceleration is zero, for then \( |\vec{a}| = \frac{d^2s}{dt^2} \mid \frac{d\vec{r}}{ds} \mid = \frac{d^2s}{dt^2} \).

3. Relative Velocity and Acceleration: Let two particles P and Q moving along the curves \( C_1 \) and \( C_2 \) have position vectors \( \vec{r}_1(t) \) and \( \vec{r}_2(t) \) at time \( t \), so that \( \vec{r}(t) = \vec{PQ} = \vec{r}_2(t) - \vec{r}_1(t) \)

   Differentiating w. r. t. \( t \), \( \frac{d\vec{r}}{dt} = \frac{d\vec{r}_2}{dt} - \frac{d\vec{r}_1}{dt} \quad \ldots (1) \)

   This defines the relative velocity of Q w. r. t. P and states that the velocity of Q relative to P = Velocity vector of Q – Velocity vector of P.

   Again differentiating (1) w. r. t. \( t \), we have

   \( \frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}_2}{dt^2} - \frac{d^2\vec{r}_1}{dt^2} \)

   This defines the relative acceleration of Q w. r. t. and states that Acceleration of Q relative to P = Acceleration of Q – Acceleration of P.
Example 11: Find the tangential and normal acceleration of a particle moving in a plane curve in Cartesian coordinates.

Solution: Let \( \vec{r} \) be the position vector the point P, a function of a scalar t. In particular, if the scalar variable t is taken as an arc length s along the curve C measured from some fixed point, that is, \( x = x(s), \ y = y(s), \ z = z(s) \) then \( \vec{r} = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k} \)

So that \( \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} \) \( \ldots (1) \)

And \( \left[ \frac{d\vec{r}}{ds} \right]^2 = \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 + \left( \frac{dz}{ds} \right)^2 \) \( \ldots (2) \)

For two dimension curves we have in calculus

\[ (ds)^2 = (dx)^2 + (dy)^2 \]

which when extended to the space, becomes

\[ (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \]

Or \( \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 + \left( \frac{dz}{ds} \right)^2 = 1 \)

Therefore (2) gives \( \left[ \frac{d\vec{r}}{ds} \right]^2 = 1 \)

That means, \( \frac{d\vec{r}}{ds} \) is a unit vector along the tangent and (1) represents a unit tangent vector along the curve C in space.

Therefore, Velocity \( \vec{v} \) of the particle at any point of the curve is given by

\[ \vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \frac{ds}{dt} = \vec{T} \] \( \ldots (3) \)

where \( v = \frac{ds}{dt} \) and \( \vec{T} = \frac{d\vec{r}}{ds} \) is the unit vector along the tangent.

Thus \( v = \frac{ds}{dt} \) is the tangential component of the velocity and the normal component of the velocity is zero.

Next, acceleration \( \vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\vec{T})}{dt} = \frac{dv}{dt} \vec{T} + v \frac{d\vec{T}}{dt} \)

or \( \vec{a} = \frac{d^2s}{dt^2} \vec{T} + \frac{ds}{dt} \frac{d\vec{T}}{ds} \frac{ds}{dt} = \frac{dv}{dt} \vec{T} + v^2 \frac{d\vec{T}}{d\psi} \frac{d\psi}{ds} \frac{ds}{dt} \)

\[ = \frac{dv}{dt} \vec{T} + \frac{v^2}{\rho} \frac{d\psi}{ds} \frac{ds}{dt} \] \( \text{since radius of curvature, } \rho = \frac{d\psi}{ds} \) \( \ldots (4) \)

From the adjoining figure, \( \vec{T} = \overrightarrow{PQ} \) is along the tangent at P to the curve C and \( \vec{N} \) is the unit vector along the normal to P.

\[ \vec{T} = \cos \psi \hat{i} + \sin \psi \hat{j} \quad \text{and} \quad \vec{N} = \cos \left( \frac{\pi}{2} + \psi \right) \hat{i} + \sin \left( \frac{\pi}{2} + \psi \right) \hat{j} = -\sin \psi \hat{i} + \cos \psi \hat{j} \]

Now \( \frac{d\vec{T}}{d\psi} = -\sin \psi \hat{i} + \cos \psi \hat{j} = \vec{N} \)

Therefore, equation (4) becomes \( \vec{a} = \frac{dv}{dt} \vec{T} + \frac{v^2}{\rho} \vec{N} \)

Which shows that tangential and normal components of acceleration at the point P are \( \frac{dv}{dt} \) and \( \frac{v^2}{\rho} \).

Since \( \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds} \) so the tangential component of acceleration is also written as \( v \frac{dv}{ds} \).
Example 12: Find the radial and transverse acceleration of a particle moving in a plane curve in Polar coordinates.

**Solution:** Let the position vector of a moving particle \( P(r, \theta) \) be \( \mathbf{r} \) so that 
\[
\mathbf{r} = r \hat{\mathbf{r}} = r (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})
\]
at any time \( t \).

Then the velocity of the particle is 
\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt} \hat{\mathbf{r}} + r \frac{d\theta}{dt} \hat{\mathbf{u}}
\]

As \( \hat{\mathbf{r}} = (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \) so 
\[
\frac{d\mathbf{r}}{dt} = \left[ \frac{dr}{dt} \hat{\mathbf{r}} \right] + \left[ r \frac{d\theta}{dt} \hat{\mathbf{u}} \right] = \hat{\mathbf{r}} + r \frac{d\theta}{dt} \hat{\mathbf{u}}
\]

Therefore, \( \frac{d\mathbf{r}}{dt} \) is perpendicular to \( \mathbf{r} \) and \( \frac{d\mathbf{r}}{dt} = \frac{d\theta}{dt} \hat{\mathbf{u}} \)

So the radial and transverse components of the velocity are \( \frac{dr}{dt} \) and \( r \frac{d\theta}{dt} \).

Also 
\[
\mathbf{a} = \frac{d^2 \mathbf{r}}{dt^2} = \left[ \frac{d^2 r}{dt^2} \hat{\mathbf{r}} \right] + \left[ \frac{dr}{dt} \frac{d\theta}{dt} \hat{\mathbf{u}} \right] = \hat{\mathbf{r}} + \frac{dr}{dt} \hat{\mathbf{u}}
\]

(\( \mathbf{u} = (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \) gives \( \frac{du}{dt} = -\frac{d\theta}{dt} \hat{\mathbf{r}} \))

Thus radial and transverse components of the acceleration are 
\[
\left[ \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] \hat{\mathbf{r}} + \left[ \frac{2 \frac{dr}{dt} \frac{d\theta}{dt} \hat{\mathbf{u}} + \frac{r^2 \frac{d^2 \theta}{dt^2}}{dt} \right] \hat{\mathbf{u}}
\]

Example 13: A particle moves along the curve 
\[
\mathbf{r} = (t^3 - 4t) \mathbf{i} + (t^2 + 4t) \mathbf{j} + (8t^2 - 3t^3) \mathbf{k}
\]

where \( t \) denotes the time. Find the magnitudes of acceleration along the tangent and normal at time \( t = 2 \).

**Solution:** The velocity of the particle is 
\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = (3t^2 - 4) \mathbf{i} + (2t + 4) \mathbf{j} + (16t - 9t^2) \mathbf{k}
\]

And the acceleration is 
\[
\mathbf{a} = \frac{d^2 \mathbf{r}}{dt^2} = (6t) \mathbf{i} + (2) \mathbf{j} + (16 - 18t) \mathbf{k}
\]

At \( t = 2 \), \( \mathbf{v} = 8 \mathbf{i} + 8 \mathbf{j} - 4 \mathbf{k} \) and \( \mathbf{a} = 12 \mathbf{i} + 2 \mathbf{j} - 20 \mathbf{k} \)

Since the velocity vector is also the tangent vector to the curve, so the magnitude of acceleration along the tangent at \( t = 2 \) is 
\[
|\mathbf{a}| = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{(12 \mathbf{i} + 2 \mathbf{j} - 20 \mathbf{k}) \cdot (8 \mathbf{i} + 8 \mathbf{j} - 4 \mathbf{k})}{\sqrt{64+64+16}} = \frac{16}{12} = 16
\]

And, the magnitude of acceleration along the normal at \( t = 2 \) is 
\[
|\mathbf{a} - \text{Component of } \mathbf{a} \text{ along the tangent at } t = 2| = \left| (12 \mathbf{i} + 2 \mathbf{j} - 20 \mathbf{k}) - \frac{16(8 \mathbf{i} + 8 \mathbf{j} - 4 \mathbf{k})}{12} \right| = \frac{4 \mathbf{i} - 26 \mathbf{j} - 44 \mathbf{k}}{3} = 2\sqrt{73}
\]

Example 14: A person going east wards with a velocity of 4 km per hour, finds that the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.
Solution: Let the actual velocity of the wind is $\mathbf{v} = xi + yj$, where $i$ and $j$ represent velocities of 1 km per hour towards the east and north respectively. As the person is going eastwards with a velocity of 4 km per hour, his actual velocity is $4i$. Then the velocity of the wind relative to the man is $(x\mathbf{i} + y\mathbf{j}) - 4\mathbf{i}$, which is parallel to $-\mathbf{j}$, as it appears to blow from the north. Hence $x = 4$. When the velocity of the person becomes $8\mathbf{i}$, the velocity of the wind relative to a man is $x\mathbf{i} + y\mathbf{j} - 8\mathbf{i}$. But this is parallel to $-\mathbf{i} + \mathbf{j}$. ∴ $x - 8y = 1$ which gives $y = -4$ (using (1)) Hence the actual velocity of the wind is $4\mathbf{i} - 4\mathbf{j}$ i.e. $4\sqrt{2}$ km per hour towards south east.

Example 15: A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 3$ where $t$ is the time. Find the components of velocity and acceleration at $t = 1$ in the direction of the vector $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

Solution: Let $\mathbf{r}$ be the position vector of the particle at any time $t$,
then $\mathbf{r} = (t^3 + 1)\mathbf{i} + (t^2)\mathbf{j} + (2t + 3)\mathbf{k}$
So the velocity is $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (3t^2)\mathbf{i} + (2t)\mathbf{j} + (2)\mathbf{k}$
And the acceleration is $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (6t)\mathbf{i} + (2)\mathbf{j} + (0)\mathbf{k}$
At $t = 1$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$
Also the unit vector in the direction of the given vector $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ is $= \frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{|\mathbf{i} + \mathbf{j} + 3\mathbf{k}|} = \frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{11}}$
Now the component of velocity at $t = 1$, in the direction of the vector $\mathbf{i} + \mathbf{j} + 3\mathbf{k} = \frac{(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 3\mathbf{k})}{\sqrt{11}} = \frac{3 + 2 + 6}{\sqrt{11}} = \frac{11}{\sqrt{11}}$
And the component of acceleration at $t = 1$, in the direction of the vector $\mathbf{i} + \mathbf{j} + 3\mathbf{k} = \frac{(6\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 3\mathbf{k})}{\sqrt{11}} = \frac{6 + 2 + 0}{\sqrt{11}} = \frac{8}{\sqrt{11}}$

ASSIGNMENT 2

1. The particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where $t$ is the time variable. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.
2. A particle (with position vector $\mathbf{r}$) is moving in a circle with constant angular velocity $\omega$. Show by vector methods, that the acceleration is equal to $-\omega^2 \mathbf{r}$.
3. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where $t$ is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.
4. The position vector of a particle at time $t$ is $\mathbf{r} = \cos(t - 1)\mathbf{i} + \sinh(t - 1)\mathbf{j} + at^3\mathbf{k}$. Find the condition imposed on $a$ by requiring that at time $t = 1$, the acceleration is normal to the position vector.
5. A particle moves so that its position vector is given by $\mathbf{r} = \cos \omega t \hat{i} + \sin \omega t \hat{j}$. Show that the velocity $\mathbf{v}$ of the particle is perpendicular to $\mathbf{r}$ and $\mathbf{r} \times \mathbf{v}$ is a constant vector.

6. A particle moves along a catenary $s = c \tan \psi$. The direction of acceleration at any point makes equal angles with the tangent and normal to the path at that point. If the speed at vertex ($\psi = 0$) be $v_0$, show that the magnitude of velocity and acceleration at any point are given by $v_0 e^\psi$ and $\frac{\sqrt{2} v_0^2 e^{2\psi} \cos^2 \psi}{c}$ respectively.

7. The position vector of a moving particle at a time $t$ is $\mathbf{r} = t^2 \hat{i} - t^3 \hat{j} + t^4 \hat{k}$. Find the tangential and normal components of acceleration at $t = 1$.

8. A vessel A is a sailing with a velocity of 11 knots per hour in the direction south-east and a second vessel B is sailing with a velocity of 13 knots per hour in a direction 30° of north. Find the velocity of A relative to B.

9. The velocity of a boat relative to water is represented by $3\hat{i} + 4\hat{j}$ and that of water relative to earth is $\hat{i} - 3\hat{j}$. What is the velocity of the boat relative to earth if $\hat{i}$ and $\hat{j}$ represent one KM an hour east and north respectively?

10. A person travelling towards the north-east with a velocity of 6 KM per hour finds that the wind appears to blow from the north, but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1} \frac{1}{2}$ to the north of east. Show that the actual velocity of the wind is $3\sqrt{2}$ KM per hour towards the east.

17.5 DEL APPLIED TO SCALAR POINT FUNCTIONS: GRADIENT [KUK 2009]

Del Operator: Del operator is a vector differential operator and is written as

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

Gradient of a Scalar Function: Let $\phi(x, y, z)$ be a scalar function of three variables defined over a region $R$ of space. Then gradient of $\phi$ is a vector function defined as

$$\phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z},$$

wherever the partial derivatives exists. It may also be denoted as $\text{grad}(\phi)$. The del operator is also called the gradient operator.

Level Surface: Let $\phi(x, y, z)$ be a scalar valued function and $C$ is a constant. The surface given by $\phi(x, y, z) = C$ through a point $P(\vec{r})$ is such that at each point on it the function has same value, is called the level surface of $\phi(x, y, z)$ through $P$, e.g. equi-potential or isothermal surfaces.

In other words, locus of the point $P(\vec{r})$ satisfying $\phi(\vec{r}) = C$ form a surface through $P$. This surface is called the level surface through $P$.

Gradient as Normal or Geometrical Interpretation of Gradient
Let $\emptyset(\vec{r}) = \emptyset(x,y,z)$ is a scalar function where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. And $\emptyset(x,y,z) = C$ is the level surface of $\emptyset$ through $P(\vec{r})$. Let $Q(\vec{r} + \delta \vec{r})$ be a point on neighboring level surface $\emptyset + \delta \emptyset$, then

$$\nabla \emptyset \cdot \delta \vec{r} = \left( i \frac{\partial \emptyset}{\partial x} + j \frac{\partial \emptyset}{\partial y} + k \frac{\partial \emptyset}{\partial z} \right) \cdot (\delta x\vec{i} + \delta y\vec{j} + \delta z\vec{k})$$

$$= \frac{\partial \emptyset}{\partial x} \delta x + \frac{\partial \emptyset}{\partial y} \delta y + \frac{\partial \emptyset}{\partial z} \delta z = \delta \emptyset \quad \cdots (1)$$

Now if $P$ and $Q$ lie on same level surface i.e. $\emptyset$ & $\emptyset + \delta \emptyset$ are same, then $\delta \emptyset = 0$.

Implies $\nabla \emptyset \cdot \delta \vec{r} = 0$ \hspace{2cm} (using (1))

Therefore, $\nabla \emptyset$ is perpendicular (normal) to every $\delta \vec{r}$ lying on this surface.

Hence, $\nabla \emptyset$ is normal to the surface $\emptyset(x,y,z) = C$ and we can write $\nabla \emptyset = |\nabla \emptyset| \hat{n}$, where $\hat{n}$ is unit normal vector to the surface.

See the Fig. 17.7, if the perpendicular distance $PM$ between the surfaces through $P$ and $Q$ be $\delta n$, then rate of change of $\emptyset$ along the normal to the surface through $P$ is

$$\frac{\partial \emptyset}{\partial n} = \lim_{\delta r \to 0} \frac{\delta \emptyset}{\delta n} = \lim_{\delta n \to 0} \frac{\nabla \emptyset \cdot \delta \vec{r}}{\delta n} \quad \cdots (2)$$

$$= |\nabla \emptyset| \lim_{\delta n \to 0} \frac{\delta n}{\delta n} = |\nabla \emptyset| \quad \text{As} \ \hat{n} \cdot \delta \vec{r} = |\delta \vec{r}| \cos \theta = \delta n$$

Hence the magnitude of $\nabla \emptyset$ i.e. $|\nabla \emptyset| = \frac{\partial \emptyset}{\partial n}$ \hspace{2cm} \cdots (3)

Thus $\nabla \emptyset$ is normal vector to the level surface $\emptyset(x,y,z) = C$ and its magnitude represents the rate of change of $\emptyset$ along this normal.

**Directional Derivative:** If $\delta r$ denotes the length $PQ$ and $\hat{u}$ be the unit vector in the direction of $PQ$, the limiting value of $\frac{\delta f}{\delta r}$ as $\delta r \to 0$ (i.e. $\frac{\delta f}{\delta r}$) is known as the **directional derivative** of $f$ along the direction $PQ$.

Since $\delta r = \frac{\delta n}{\cos \alpha} = \frac{\delta \vec{n}}{\hat{n} \cdot \hat{u}}$, therefore $\frac{\delta f}{\delta r} = \lim_{\delta r \to 0} \left[ \hat{n} \cdot \hat{u} \frac{\delta f}{\delta n} \right] = \hat{u} \cdot \frac{\delta f}{\delta n} \hat{n} = \hat{u} \cdot \nabla f$

Thus the directional derivative of $f$ in the direction of $\hat{u}$ is the resolved part of $\nabla f$ in the direction of $\hat{u}$.

Since $\nabla f \cdot \hat{u} = |\nabla f| \cos \alpha \leq |\nabla f|$  

It follows that $\nabla f$ gives the maximum rate of change of $f$.

**Properties of Gradient Operator:** Let $\emptyset(x,y,z) \& \psi(x,y,z)$ are two differentiable scalar functions defined over some region $R$. then the gradient operator has following properties:
(i) Gradient of a constant multiple of scalar function $\emptyset$

$$\nabla(C\emptyset) = C \nabla\emptyset \quad \text{or} \quad \nabla(C\emptyset) = C \nabla\emptyset$$

Proof: Consider $\nabla(C\emptyset) = \nabla(C\emptyset)$

$$= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (C\emptyset)$$

$$= \hat{i} \frac{\partial C}{\partial x} \emptyset + \hat{j} \frac{\partial C}{\partial y} \emptyset + \hat{k} \frac{\partial C}{\partial z} \emptyset$$

$$= C \left( \hat{i} \frac{\partial \emptyset}{\partial x} + \hat{j} \frac{\partial \emptyset}{\partial y} + \hat{k} \frac{\partial \emptyset}{\partial z} \right)$$

$$= C \left( \hat{i} \frac{\partial \emptyset}{\partial x} + \hat{j} \frac{\partial \emptyset}{\partial y} + \hat{k} \frac{\partial \emptyset}{\partial z} \right) \emptyset$$

$$= C \nabla \emptyset = C \nabla \emptyset$$

Hence $\nabla(C\emptyset) = C \nabla\emptyset$.

(ii) Gradient of sum or difference of two scalar functions

$$\nabla(\emptyset \pm \psi) = \nabla(\emptyset) \pm \nabla\psi \quad \text{or} \quad \nabla(\emptyset \pm \psi) = \nabla\emptyset \pm \nabla\psi$$

Proof: Consider $\nabla(\emptyset \pm \psi) = \nabla(\emptyset \pm \psi)$

$$= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (\emptyset \pm \psi)$$

$$= \hat{i} \frac{\partial \emptyset}{\partial x} \pm \hat{i} \frac{\partial \psi}{\partial x} + \hat{j} \frac{\partial \emptyset}{\partial y} \pm \hat{j} \frac{\partial \psi}{\partial y} + \hat{k} \frac{\partial \emptyset}{\partial z} \pm \hat{k} \frac{\partial \psi}{\partial z}$$

$$= \left( \hat{i} \frac{\partial \emptyset}{\partial x} + \hat{j} \frac{\partial \emptyset}{\partial y} + \hat{k} \frac{\partial \emptyset}{\partial z} \right) \emptyset \pm \left( \hat{i} \frac{\partial \psi}{\partial x} + \hat{j} \frac{\partial \psi}{\partial y} + \hat{k} \frac{\partial \psi}{\partial z} \right) \psi$$

$$= \nabla \emptyset \pm \nabla\psi = \nabla\emptyset \pm \nabla\psi$$

Hence $\nabla(\emptyset \pm \psi) = \nabla\emptyset \pm \nabla\psi$.

(iii) Gradient of product of two scalar functions

$$\nabla(\emptyset \psi) = \emptyset \nabla\psi + \psi \nabla\emptyset \quad \text{or} \quad \nabla\emptyset \psi = \emptyset \nabla\psi + \psi \nabla\emptyset$$

Proof: Consider $\nabla(\emptyset \psi) = \nabla(\emptyset \psi)$

$$= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (\emptyset \psi)$$

$$= \hat{i} \frac{\partial \emptyset}{\partial x} (\emptyset \psi) + \hat{j} \frac{\partial \emptyset}{\partial y} (\emptyset \psi) + \hat{k} \frac{\partial \emptyset}{\partial z} (\emptyset \psi)$$

17
\[
\begin{align*}
&= i \left( \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x} \right) + j \left( \frac{\partial \psi}{\partial y} + \psi \frac{\partial \phi}{\partial y} \right) + k \left( \frac{\partial \psi}{\partial z} + \psi \frac{\partial \phi}{\partial z} \right) \\
&= \varnothing \left( i \frac{\partial \psi}{\partial x} + j \frac{\partial \psi}{\partial y} + k \frac{\partial \psi}{\partial z} \right) + \psi \left( i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) \\
&= \varnothing \nabla \psi + \psi \nabla \varnothing = \varnothing \text{ grad}(\psi) \pm \psi \text{ grad}(\varnothing).
\end{align*}
\]

Hence \( \text{ grad}(\varnothing \psi) = \varnothing \text{ grad}(\psi) \pm \psi \text{ grad}(\varnothing) \).

(iv) Gradient of quotient of two scalar functions

\[

\text{ grad} \left( \frac{\phi}{\psi} \right) = \frac{\psi \text{ grad} (\varnothing) - \varnothing \text{ grad} (\psi)}{(\psi)^2} \quad \text{or} \quad \nabla \left( \frac{\phi}{\psi} \right) = \frac{\psi \nabla (\varnothing) - \varnothing \nabla (\psi)}{(\psi)^2} \quad \text{provided} \ \psi \neq 0.
\]

Proof: Consider \( \text{ grad} \left( \frac{\phi}{\psi} \right) = \nabla \left( \frac{\phi}{\psi} \right) \)

\[
\begin{align*}
&= \frac{i}{\psi}\left( \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x} \right) + \frac{j}{\psi}\left( \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y} \right) + \frac{k}{\psi}\left( \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z} \right) \\
&= \frac{1}{\psi^2} \left[ \psi \left( i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) - \varnothing \left( i \frac{\partial \psi}{\partial x} + j \frac{\partial \psi}{\partial y} + k \frac{\partial \psi}{\partial z} \right) \right] \\
&= \frac{1}{\psi^2} \left[ \psi \text{ grad}(\varnothing) - \varnothing \text{ grad}(\psi) \right] = \frac{\psi \text{ grad}(\varnothing) - \varnothing \text{ grad}(\psi)}{(\psi)^2} \\
&= \frac{\psi \nabla (\varnothing) - \varnothing \nabla (\psi)}{(\psi)^2}.
\end{align*}
\]

Hence\( \text{ grad} \left( \frac{\phi}{\psi} \right) = \frac{\psi \text{ grad} (\varnothing) - \varnothing \text{ grad} (\psi)}{(\psi)^2} \).

Example 16: If \( \varnothing = 3x^2y - y^3z^2 \), then find \( \text{ grad } \varnothing \) at \((1, -2, -1)\).

Solution: \( \text{ grad } \varnothing = \nabla (3x^2y - y^3z^2) \)

\[
\begin{align*}
&= i \frac{\partial}{\partial x}(3x^2y - y^3z^2) + j \frac{\partial}{\partial y}(3x^2y - y^3z^2) + k \frac{\partial}{\partial z}(3x^2y - y^3z^2) \\
&= i(6xy) + j(3x^2 - 3y^2z^2) + k(-2y^3z) \\
\text{At } (1, -2, -1), \quad \text{ grad } \varnothing &= (i 6 (1)(-2)) + j(3(1)^2 - 3(-2)^2(-1)^2) + k(-2(-2)^3(-1)) \\
&= -12i - 9j - 16k.
\end{align*}
\]

Example 17: Prove that \( \nabla (r^n) = n r^{n-2} \tilde{r} \), where \( \tilde{r} = xi + yj + zk \).

Solution: Here \( \tilde{r} = xi + yj + zk \) and \( r^2 = x^2 + y^2 + z^2 \)

So differentiating partially w. r. t. \( x \), \( \frac{\partial r}{\partial x} = \frac{x}{r} \), Similarly, \( \frac{\partial r}{\partial y} = \frac{y}{r} \) and \( \frac{\partial r}{\partial z} = \frac{z}{r} \) \quad \ldots \ (1)
Consider \( \nabla (r^n) = \hat{t} \frac{\partial}{\partial x} (r^n) + \hat{j} \frac{\partial}{\partial y} (r^n) + \hat{k} \frac{\partial}{\partial z} (r^n) \)

\[
= \hat{t} \left( n r^{n-1} \frac{\partial r}{\partial x} \right) + \hat{j} \left( n r^{n-1} \frac{\partial r}{\partial y} \right) + \hat{k} \left( n r^{n-1} \frac{\partial r}{\partial z} \right)
\]

\[
= \hat{t} \left( n r^{n-1} \frac{x}{r} \right) + \hat{j} \left( n r^{n-1} \frac{y}{r} \right) + \hat{k} \left( n r^{n-1} \frac{z}{r} \right)
\]

\[
= \hat{t} (n r^{n-2} x) + \hat{j} (n r^{n-2} y) + \hat{k} (n r^{n-2} z)
\]

\[
= n r^{n-2} (x \hat{t} + y \hat{j} + z \hat{k}) = n r^{n-2} \hat{r}
\]

Hence \( \nabla (r^n) = n r^{n-2} \hat{r} \)

**Example 18:** Find the unit vector normal to the surface \( x^3 + y^3 + 3xyz = 3 \) at

(i) the point \((1, 2, -1)\) **(i)** the point \((1, 3, -1)\) **(ii)** **[KUK *2006, **2011]**

**Solution:** We know that a vector normal to a surface is given by its gradient, so if \( \vec{n} \) is the vector normal to the given surface then

\[
\vec{n} = \nabla (x^3 + y^3 + 3xyz - 3) = \left( \hat{t} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^3 + y^3 + 3xyz - 3)
\]

\[
= (3x^2 + 3yz)\hat{t} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}
\]

(i) At point \((1, 2, -1)\),

\[
\vec{n} = (3(1)^2 + 3(2)(-1))\hat{t} + (3(2)^2 + 3(1)(-1))\hat{j} + (3(1)(2))\hat{k}
\]

\[
= -3\hat{t} + 9\hat{j} + 6\hat{k}
\]

Also \( |\vec{n}| = \sqrt{(-3)^2 + (9)^2 + (6)^2} = \sqrt{126} \)

Therefore the unit normal vector to the given surface at a point \((1, 2, -1)\) is

\[
\hat{n} = \frac{1}{\sqrt{126}} (-3\hat{t} + 9\hat{j} + 6\hat{k}).
\]

(ii) At point \((1, 3, -1)\),

\[
\vec{n} = (3(1)^2 + 3(3)(-1))\hat{t} + (3(3)^2 + 3(1)(-1))\hat{j} + (3(1)(3))\hat{k}
\]

\[
= -6\hat{t} + 24\hat{j} + 9\hat{k}
\]

Also \( |\vec{n}| = \sqrt{(-6)^2 + (24)^2 + (9)^2} = \sqrt{693} \)

Therefore the unit normal vector to the given surface at a point \((1, 3, -1)\) is

\[
\hat{n} = \frac{1}{\sqrt{693}} (-6 \hat{t} + 24 \hat{j} + 9 \hat{k}).
\]

**Example 19:** Show that \( \text{grad} \ e^{(x^2+y^2+z^2)} = 2e r^2 \), where \( r^2 = |\vec{r}|^2 = x^2 + y^2 + z^2 \).

**Solution:** Here \( r^2 = x^2 + y^2 + z^2 \)

So differentiating w. r. t. \( x \), \( \frac{\partial r}{\partial x} = \frac{x}{r} \). Similarly, \( \frac{\partial r}{\partial y} = \frac{y}{r} \) and \( \frac{\partial r}{\partial z} = \frac{z}{r} \)

Now \( \text{grad} \ e^{(x^2+y^2+z^2)} = \nabla e^{r^2} = \hat{t} \frac{\partial}{\partial x} (e^{r^2}) + \hat{j} \frac{\partial}{\partial y} (e^{r^2}) + \hat{k} \frac{\partial}{\partial z} (e^{r^2}) \)

\[
= \hat{t} \left( e^{r^2} 2r \frac{\partial r}{\partial x} \right) + \hat{j} \left( e^{r^2} 2r \frac{\partial r}{\partial y} \right) + \hat{k} \left( e^{r^2} 2r \frac{\partial r}{\partial z} \right)
\]
\[ e^r \frac{d}{dr} 2r \left( l \frac{dr}{dx} + j \frac{dr}{dy} + k \frac{dr}{dz} \right) = e^r \frac{d}{dr} 2r \left( \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j} + \frac{z}{r} \hat{k} \right) \]
\[ = 2e^r \frac{d}{dr} \hat{r} \]

**Example 20:** If \( u = x + y + z \), \( v = x^2 + y^2 + z^2 \) and \( w = yz + zx + xy \), then show that \( \text{grad} \ u \), \( \text{grad} \ v \) and \( \text{grad} \ w \) are coplanar.

**Solution:** Consider \( \text{grad} \ u = \frac{\partial u}{\partial x} + j \frac{\partial u}{\partial y} + k \frac{\partial u}{\partial z} = \hat{i}(1) + j(1) + k(1) = \hat{i} + j + k \)

\[ \text{grad} \ v = \frac{\partial v}{\partial x} + j \frac{\partial v}{\partial y} + k \frac{\partial v}{\partial z} = \hat{i}(2x) + j(2y) + k(2z) = 2x \hat{i} + 2y \hat{j} + 2z \hat{k} \]

\[ \text{grad} \ w = \frac{\partial w}{\partial x} + j \frac{\partial w}{\partial y} + k \frac{\partial w}{\partial z} = (y + z) \hat{i} + (z + x) \hat{j} + (x + y) \hat{k} \]

We know that three vectors \( \hat{a}, \hat{b}, \text{and} \ \hat{c} \) are coplanar if their scalar triple product is zero \( i.e. \quad [\hat{a} \ \hat{b} \ \hat{c}] = 0 \)

Consider \[ [\text{grad} \ u \ \ \text{grad} \ v \ \ \text{grad} \ w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y + z & z + x & x + y \end{vmatrix} \]
\[ = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y + z & z + x & x + y \end{vmatrix} \] (taking common 2 from \( R_2 \))
\[ = 2 \begin{vmatrix} x + y + z & x + y + z & x + y + z \\ y + z & z + x & z + x + y \end{vmatrix} \] (adding \( R_2 \) and \( R_3 \))
\[ = 2(x + y + z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y + z & z + x & x + y \end{vmatrix} \] (taking common \( x + y + z \) from \( R_2 \))
\[ = 2(x + y + z)(0) = 0 \]

Hence \( \text{grad} \ u \), \( \text{grad} \ v \) and \( \text{grad} \ w \) are coplanar.

**Example 21:** Show that \( \text{grad} \ f(r) \times \hat{r} = \vec{0} \).

**Solution:** Here \( \text{grad} \ f(r) = \nabla f(r) = \frac{\partial}{\partial x}(f(r)) + j \frac{\partial}{\partial y}(f(r)) + k \frac{\partial}{\partial z}(f(r)) \)
\[ = \hat{i} \left( f'(r) \frac{dr}{dx} \right) + j \left( f'(r) \frac{dr}{dy} \right) + k \left( f'(r) \frac{dr}{dz} \right) \]
\[ = f'(r) \left( \hat{i} \frac{dr}{dx} + j \frac{dr}{dy} + k \frac{dr}{dz} \right) = f'(r) \left( \hat{i} \frac{x}{r} + j \frac{y}{r} + k \frac{z}{r} \right) \]
\[ = f'(r) \frac{\hat{r}}{r} \]
Now \( \text{grad } f(r) \times \vec{r} = f'(r) \frac{\vec{r}}{r} \times \vec{r} = f'(r) \frac{1}{r} (\vec{r} \times \vec{r}) = 0 \)  
\[ \text{since } (\vec{r} \times \vec{r}) = 0 \]

**Example 22:** Find the directional derivative of \( f(x, y, z) = x^2y^2z^2 \) at the point \((1, 1, -1)\) in the direction of the tangent to the curve \( x = e^t, y = 2 \sin t + 1, z = t - \cos t \) at \( t = 0 \).

**Solution:** Consider \( \nabla f(x, y, z) = \nabla (x^2y^2z^2) = (2xy^2z^2)\hat{i} + (2yx^2z^2)\hat{j} + (2z^2y^2)\hat{k} \)
At \((1, 1, -1)\), \( \nabla f(x, y, z) = (2 \hat{i} + 2 \hat{j} - 2 \hat{k}) \)

Now \( \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (e^t)\hat{i} + (2 \sin t + 1)\hat{j} + (t - \cos t)\hat{k} \)
So tangent to the curve is \( \frac{d\vec{r}}{dt} = (e^t)\hat{i} + (2 \cos t)\hat{j} + (1 + \sin t)\hat{k} \)
At \( t = 0 \), \( \frac{d\vec{r}}{dt} = \hat{i} + 2 \hat{j} + \hat{k} \)
And the unit tangent vector is \( \frac{d\vec{r}}{dt} / \left| \frac{d\vec{r}}{dt} \right| = \frac{\hat{i} + 2 \hat{j} + \hat{k}}{\sqrt{6}} \)
So the required directional derivative in the direction of the tangent is \( \nabla f(x, y, z) \cdot \left( \frac{d\vec{r}}{dt} / \left| \frac{d\vec{r}}{dt} \right| \right) = (2 \hat{i} + 2 \hat{j} - 2 \hat{k}) \cdot (\hat{i} + 2 \hat{j} + \hat{k}) / \sqrt{6} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{3}}{3} \)

**Example 23:** If the directional derivative \( \emptyset = a x^2 y + b y^2 z + c z^2 x \) at the point \((1, 1, 1)\) has maximum magnitude 15 in the direction parallel to the line \( \frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \), find the values of \( a, b \) and \( c \).  

**Solution:** Consider \( \nabla \emptyset = \nabla (a x^2y + b y^2z + c z^2x) = (2ax y + c z^2)\hat{i} + (a x^2 + 2b yz)\hat{j} + (b y^2 + 2c zx)\hat{k} \)
At \((1, 1, 1)\), \( \nabla \emptyset = (2a + c)\hat{i} + (a + 2b)\hat{j} + (b + 2c)\hat{k} \)
We know that directional derivative of \( \emptyset \) is maximum in the direction of its normal vector \( \nabla \emptyset \), but it is given to be maximum in the direction of the line \( \frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \).
Therefore, the line and normal vector are parallel to each other, which results as:
\[
\frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1} \quad \ldots (1)
\]
Taking first two members of \((1)\), \( 3a + 2b + c = 0 \)
and by last two members of \((1)\), \( a + 4b + 4c = 0 \)
Solving the two obtained equations, \( \frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = \lambda (\text{Let}) \)
\[=> \quad a = 4\lambda, b = -11\lambda \text{ and } c = 10\lambda \quad \ldots (2) \]
Also given that maximum magnitude of directional derivative is 15 units \( i.e. |\nabla \emptyset| = 15 \)
So, \( (2a + c)^2 + (a + 2b)^2 + (b + 2c)^2 = (15)^2 \quad \ldots (3) \)
Putting the values of \( a, b \) and \( c \) from \((2)\),
\[(8\lambda + 10\lambda)^2 + (4\lambda - 22\lambda)^2 + (-11\lambda + 20\lambda)^2 = (15)^2 \Rightarrow \lambda = \pm \frac{5}{9} \]

Hence \(a = \pm \frac{20}{9}, b = \mp \frac{55}{9}, c = \pm \frac{50}{9}.\)

**Example 24:** In what direction from \((3, 1, -2)\) is the directional derivative of \(\phi = x^2y^2z^4\) maximum? Find also the magnitude of this maximum.  

**Solution:** The vector normal to the given surface is
\[\vec{n} = \nabla \phi = \nabla (x^2y^2z^4) = \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^2y^2z^4)\]
\[= 2xy^2z^4 \hat{i} + 2x^2yz^4 \hat{j} + 4x^2y^2z^3 \hat{k}\]

At point \((3, 1, -2)\)
\[\vec{n} = 2(3)(1)^2(-2)^4 \hat{i} + 2(3)^2(1)(-2)^4 \hat{j} + 4(3)^2(1)^2(-2)^3 \hat{k}\]
\[= 96 \hat{i} + 288 \hat{j} - 288 \hat{k}\]
Also \(|\vec{n}| = \sqrt{(96)^2 + (288)^2 + (-288)^2} = 96\sqrt{19}\)

So the directional derivative of given surface will be maximum in the direction of \(96 \hat{i} + 288 \hat{j} - 288 \hat{k}\) and the magnitude of this maximum is \(96\sqrt{19}\).

**Example 25:** Find the angle between the surfaces \(x^2 + y^2 + z^2 = 9\) and \(z = x^2 + y^2 - 3\) at \((2, -1, 2)\).

**Solution:** Given surfaces are
\[\phi_1 = x^2 + y^2 + z^2 - 9 = 0 \quad \ldots \quad (1)\]
and \[\phi_2 = x^2 + y^2 - z - 3 = 0 \quad \ldots \quad (2)\]

We know that gradient of a surface gives the vector normal to the surface. Let \(\vec{n}_1\) and \(\vec{n}_2\) are the vectors normal to the surfaces \((1)\) and \((2)\) respectively.

Now \[\nabla \phi_1 = \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (x^2 + y^2 + z^2 = 9) = 2x \hat{i} + 2y \hat{j} + 2z \hat{k} \quad \ldots \quad (3)\]
\[\nabla \phi_2 = \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (x^2 + y^2 - z - 3) = 2x \hat{i} + 2y \hat{j} - \hat{k} \quad \ldots \quad (4)\]

So, \(\vec{n}_1 = (\nabla \phi_1)_{at \,(2,-1,2)} = 4 \hat{i} - 2 \hat{j} + 4 \hat{k}\) and \(\vec{n}_2 = (\nabla \phi_2)_{at \,(2,-1,2)} = 4 \hat{i} - 2 \hat{j} - \hat{k}\)

Let \(\theta\) be the angle between the given surfaces at point \((2, -1, 2)\), then \(\theta\) will also be an angle between their normals \(\vec{n}_1\) and \(\vec{n}_2\).

Therefore, \[\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(4 \hat{i} - 2 \hat{j} + 4 \hat{k}) (4 \hat{i} - 2 \hat{j} - \hat{k})}{4 \hat{i} - 2 \hat{j} + 4 \hat{k} || 4 \hat{i} - 2 \hat{j} - \hat{k}|} = \frac{16+4-4}{36\sqrt{21}} = \frac{8}{3\sqrt{21}}\]

**Example 26:** Find the constants \(a\) and \(b\) so that the surface \(ax^2 - byz = (a + 2)x\) is orthogonal to the surface \(4x^2y + z^3 = 4\) at the point \((1, -1, 2)\).

**Solution:** Given surfaces are
\[f = ax^2 - byz - (a + 2)x = 0 \quad \ldots \quad (1)\]
and \[g = 4x^2y + z^3 - 4 = 0 \quad \ldots \quad (2)\]

\[\frac{\partial f}{\partial x} = 2ax - (a + 2) \quad \frac{\partial f}{\partial y} = -bz \quad \frac{\partial f}{\partial z} = -by \]
\[\frac{\partial g}{\partial x} = 8xy \quad \frac{\partial g}{\partial y} = 4x^2 \quad \frac{\partial g}{\partial z} = 3z^2 \]

At point \((1, -1, 2)\)
\[\frac{\partial f}{\partial x} = 2a - (a + 2) = 0 \quad \Rightarrow \quad a = 2\]
\[\frac{\partial g}{\partial x} = 8 \quad \Rightarrow \quad b = 2\]

Therefore, \[a = 2, b = 2\]
Let \( \vec{n}_1 \) and \( \vec{n}_2 \) are the vectors normal to the surfaces (1) and (2) at \((1, -1, 2)\), respectively.

Consider \( \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} = \hat{i} (2ax - (a + 2)) + \hat{j} (-bz) + \hat{k}(-by) \) \( \ldots (3) \)

And \( \nabla g = \hat{i} \frac{\partial g}{\partial x} + \hat{j} \frac{\partial g}{\partial y} + \hat{k} \frac{\partial g}{\partial z} = \hat{i} (8xy) + \hat{j} (4x^2) + \hat{k}(3z^2) \) \( \ldots (4) \)

So, \( \vec{n}_1 = (\nabla f)_{(1, -1, 2)} = \hat{i} (a - 2) + \hat{j} (-2b) + \hat{k}(b) = (a - 2)\hat{i} - 2b\hat{j} + b\hat{k} \)

\( \vec{n}_2 = (\nabla g)_{(1, -1, 2)} = \hat{i} (-8) + \hat{j} (4) + \hat{k}(12) = -8\hat{i} + 4\hat{j} + 12\hat{k} \)

Given that surfaces (1) and (2) cut orthogonally at \((1, -1, 2)\), so their normal vectors \( \vec{n}_1 \) and \( \vec{n}_2 \) should also be orthogonal to each other. Therefore,

\[ \vec{n}_1 \cdot \vec{n}_2 = 0 \]

\[ (a - 2)\hat{i} - 2b\hat{j} + b\hat{k} \cdot (-8\hat{i} + 4\hat{j} + 12\hat{k}) = 0 \]

\[ 2a - b = 4 \] \( \ldots (5) \)

Also the point \((1, -2, 1)\) lies on the surface (1), so we have

\[ a + 2b - (a + 2) = 0 \quad \text{or} \quad 2b - 2 = 0 \quad \text{or} \quad b = 1 \]

Putting value of \( b \) in (3), we get \[ 2a - 1 = 4 \quad \text{or} \quad a = \frac{5}{2} \]

**Example 27:** Show that the components of a vector \( \vec{r} \) along and normal (perpendicular) to a vector \( \vec{a} \), in the plane of \( \vec{r} \) and \( \vec{a} \), are \( \frac{(\vec{r} \cdot \vec{a})}{\vec{a}^2} \vec{a} \) and \( \frac{\vec{a} \times (\vec{r} \times \vec{a})}{(\vec{a}^2)^2} \).

**Solution:** Let \( \overrightarrow{OA} = \vec{a} \) and \( \overrightarrow{OB} = \vec{r} \) and \( \overrightarrow{OM} \) be the projection of \( \vec{r} \) on \( \vec{a} \) (Fig. 17.8)

\[ \vec{r} \cdot \vec{a} \]

Also the component of \( \vec{r} \) normal to \( \vec{a} = \overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM} \)

\[ = \vec{r} - \frac{(\vec{r} \cdot \vec{a})}{\vec{a}^2} \vec{a} = \frac{(\vec{a} \cdot \vec{r})\vec{a} - (\vec{r} \cdot \vec{a})\vec{a}}{\vec{a}^2} = \frac{\vec{a} \times (\vec{r} \times \vec{a})}{(\vec{a}^2)^2} \]

**Example 28:** If \( f \) and \( \vec{F} \) are point functions, prove that the components of the latter normal and tangential to the surface \( f = 0 \) are \( \frac{(\vec{F} \cdot \nabla f) \nabla f}{(\nabla f)^2} \) and \( \frac{\nabla f \times (\vec{F} \times \nabla f)}{(\nabla f)^2} \).

**Solution:** We know that for the given surface \( f = 0 \), the vector normal to the surface is given by the gradient \( \vec{n} = \nabla f \).

Now the component of \( \vec{F} \) normal to the given surface is \( \frac{(\vec{F} \cdot \nabla f) \nabla f}{|\nabla f|^2} = \frac{(\vec{F} \cdot \nabla f) \nabla f}{(\nabla f)^2} \)

And the component of \( \vec{F} \) tangential to the given surface is \( \vec{F} - \text{the normal component of } \vec{F} \)
\[ \bar{F} - \frac{(\bar{F} \cdot \nabla f)}{(\nabla f)^2} = \frac{\bar{F} (\nabla f)^2 - (\bar{F} \cdot \nabla f) (\nabla f)}{(\nabla f)^2} = \frac{\bar{F} (\nabla f)^2 - (\bar{F} \cdot \nabla f) (\nabla f)}{(\nabla f)^2} = \frac{\nabla f \times (\bar{F} \times \nabla f)}{(\nabla f)^2} \]

ASSIGNMENT 3

1. Find \( \nabla \phi \), if \( \phi = \log(x^2 + y^2 + z^2) \).

2. Show that \( \frac{1}{r} = -\frac{\bar{r}}{r^3} \).

3. What is the directional derivative of \( \phi = xy^2 + yz^3 \) at the point \((2, -1, 1)\) in the direction of the normal to the surface \( x \log z - y^2 = -4 \) at \((-1, 2, 1)\)? \[ \text{[JNTU 2005; VTU 2004]} \]

4. What is the directional derivative of \( \phi = x^2yz + 4xz^2 \) at the point \((1, -2, 1)\) in the direction of the vector \( 2\hat{i} - \hat{j} - 2\hat{k} \). \[ \text{[VTU 2007; UP Tech, JNTU 2006]} \]

5. The temperature of points in a space is given by \( T(x, y, z) = x^2 + y^2 - z \). A mosquito located at \((1, 1, 2)\) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?

6. What is the greatest rate of increase of \( u = x^2 + yz^2 \) at the point \((1, -1, 3)\)?

7. Find the angle between the tangent planes to the surfaces \( x \log z = y^2 - 1 \) and \( x^2y = 2 - z \) at the point \((1, 1, 1)\). \[ \text{[JNTU 2003]} \]

8. Calculate the angle between the normals to the surface \( xy = z^2 \) at the points \((4, 1, 2)\) and \((3, 3, -3)\).

9. Find the angle between the surfaces \( x^2 + y^2 + z^2 = 9 \) and \( x = x^2 + y^2 - 3 \) at \((2, -1, 2)\).

10. Find the values of \( \lambda \) and \( \mu \) so that the surface \( \lambda x^2y + \mu z^3 = 4 \) may cut the surface \( 5x^2 = 2yz + 9x \) orthogonally at \((1, -1, 2)\)

17.6 DEL APPLIED TO VECTOR POINT FUNCTIONS (Divergence & Curl)

1. **Divergence**: Let \( \bar{f}(x, y, z) = f_1(x, y, z)\hat{i} + f_2(x, y, z)\hat{j} + f_3(x, y, z)\hat{k} \) be a continuously differentiable vector point function. Divergence \( \bar{f}(x, y, z) \) of is a scalar which is denoted by \( \nabla \cdot \bar{f} \) and is defined as

\[
\nabla \cdot \bar{f} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.
\]

It is also denoted by \( \text{div} \bar{f} \).

**Physical interpretation of Divergence**

Consider the case of fluid flow.

Let \( \bar{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \) be the velocity of the fluid at a point \( P(x, y, z) \). Consider a small parallelepiped with edges \( \delta x, \delta y \) and \( \delta z \) parallel to the x, y and z axis.
respectively in the mass of fluid, with one of its corner at point \( P \).

So, the mass of fluid flowing in through the face \( PQRS \) per unit time = \( v_y \delta z \delta x \)

and the mass of fluid flowing out of the face \( P'Q'R'S' \) per unit time

\[
v_y + \frac{\partial v_y}{\partial y} \delta y \delta z \delta x
\]

\[
\Rightarrow \text{The net decrease in fluid mass in the parallelopiped corresponding to flow along y-axis}
\]

\[
= \left( v_y + \frac{\partial v_y}{\partial y} \right) \delta z \delta x - v_y \delta z \delta x = \frac{\partial v_y}{\partial y} \delta x \delta y \delta z
\]

Similarly, the net decrease in fluid mass in the parallelopiped corresponding to the flow along x-axis and z-axis is

\[
\frac{\partial v_x}{\partial x} \delta x \delta y \delta z \quad \text{and} \quad \frac{\partial v_z}{\partial z} \delta x \delta y \delta z
\]

respectively.

So, total decrease in mass of fluid mass in the parallelopiped per unit time

\[
= \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \delta x \delta y \delta z
\]

Thus, the rate of loss of fluid per unit volume =

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}
\]

\[
= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{i} + v_y \hat{j} + v_z \hat{k})
\]

\[
= \nabla \cdot \vec{v} = div \vec{v}
\]

Hence, \( div \vec{v} \) gives the rate at which fluid is originating or diminishing at a point per unit volume.

If the fluid is incompressible, there can be no loss or gain in the volume element i.e. \( div \vec{v} = 0 \).

Observations:

(i) if \( \vec{v} \) represent the electric flux, then \( div \vec{v} \) is the amount of flux which diverges per unit volume in unit time.

(ii) if \( \vec{v} \) represent the heat flux, then \( div \vec{v} \) is the rate at which the heat is issuing from a point per unit volume.

(iii) If the flux entering any element of space is the same as that leaving it i.e. \( div \vec{v} = 0 \) everywhere, then such a vector point function is called Solenoidal.

Example 29: If \( \vec{r} = xi + yj + zk \) then show that \( div \vec{r} = 3 \).

Solution: \( div \vec{r} = \nabla \cdot \vec{r} = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (xi + yj + zk) \)

\[
= \frac{\partial}{\partial x} (x) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z) = 1 + 1 + 1 = 3
\]

Example 30: Evaluate \( div \vec{f} \) where \( \vec{f} = 2x^2 \hat{i} - xy^2 \hat{j} + 3y^2 \hat{k} \) at \((1, 1, 1)\).

Solution: \( div \vec{f} = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (2x^2 \hat{i} - xy^2 \hat{j} + 3y^2 \hat{k}) \)

\[
= \frac{\partial}{\partial x} (2x^2) + \frac{\partial}{\partial y} (-xy^2) + \frac{\partial}{\partial z} (3y^2) = 4x - xy^2 + 6y
\]
\[
\frac{\partial}{\partial x} (2x^2z) + \frac{\partial}{\partial y} (-xy^2z) + \frac{\partial}{\partial z} (3y^2x) = 4xz - 2xyz + 0
\]

At (1, 1, 1), \( \text{div } \vec{f} = 4(1)(1) - 2(1)(1)(1) = 2 \)

**Example 31**: Determine the constant \( a \) so that the vector \( \vec{f} = (x + 3y)i + (y - 2z)j + (x + az)k \) is solenoidal.

**Solution**: Given that the vector \( \vec{f} \) is solenoidal, so \( \text{div } \vec{f} = 0 \)

\[
\left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left( (x + 3y)i + (y - 2z)j + (x + az)k \right) = 0
\]

\[
\frac{\partial}{\partial x} (x + 3y) + \frac{\partial}{\partial y} (y - 2z) + \frac{\partial}{\partial z} (x + az) = 0
\]

\[1 + 1 + a = 0 \implies a = -2\]

**2. Curl**: Let \( \vec{f}(x, y, z) = f_1(x, y, z)i + f_2(x, y, z)j + f_3(x, y, z)k \) be a continuously differentiable vector point function. Curl of \( \vec{f}(x, y, z) \) of is a vector which is denoted by \( \nabla \times \vec{f} \) and is defined as

\[
\nabla \times \vec{f} = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times \vec{f} = \hat{i} \times \frac{\partial f_3}{\partial y} - \hat{j} \times \frac{\partial f_3}{\partial x} + \hat{k} \times \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}
\]

Also in component form, curl of \( \vec{f}(x, y, z) \) is

\[
\nabla \times \vec{f} = \left| \begin{array}{ccc}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
f_1 & f_2 & f_3
\end{array} \right| = \hat{i} \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) + \hat{j} \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) + \hat{k} \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)
\]

**Physical Interpretation of Curl**

Consider the motion of a rigid body rotating with angular velocity \( \vec{\omega} \) about an axis OA, where O is a fixed point in the body. Let \( \vec{r} \) be the position vector of any point P of the body. The point P describing a circle whose center is M and radius is PM = r sin \( \theta \) where \( \theta \) is the angle between \( \vec{\omega} \) and \( \vec{r} \), then the velocity of P is \( \vec{v} = \omega r \sin \theta \vec{n} = \vec{\omega} \times \vec{r} \). So, if \( \vec{v} \) is the linear velocity of P, then \( \vec{v} = \omega r \sin \theta \vec{n} = \vec{\omega} \times \vec{r} \)
Now, if $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ and $\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$, then

$$\text{Curl } \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

And

$$\text{Curl } \vec{v} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix} = \hat{i}(\omega_1 + \omega_1) + \hat{j}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3) = 2\omega_1 \hat{i} + 2\omega_2 \hat{j} + 2\omega_3 \hat{k} = 2 \vec{\omega}$$

Hence $\vec{\omega} = \frac{1}{2} \text{Curl } \vec{v}$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector.

**Observations:**

(i) The curl of a vector point function gives the measure of the angular velocity at a point.

(ii) If the curl of a vector point function becomes zero i.e. $\nabla \times \vec{f} = 0$, then $\vec{f}$ is called an irrotational vector.

**Example 32:** If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that $\text{curl } \vec{r} = \vec{0}$.

**Solution:**

$$\text{curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \hat{i}(0 - 0) + \hat{j}(0 - 0) + \hat{k}(0 - 0) = \vec{0}$$

**Example 33:** Find $a$ so that the vector $\vec{f} = (axy - z^3) \hat{i} + (a - 2)x^2 \hat{j} + (1 - a)xz^2 \hat{k}$ is irrotational.

**Solution:**

Given that $\vec{f}$ is irrotational, therefore $\text{curl } \vec{f} = \vec{0}$ \hspace{1cm} \ldots (1)

But $\text{curl } \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a - 2)x^2 & (1 - a)xz^2 \end{vmatrix} = \hat{i}(0 - 0) + \hat{j}(3z^2 - (1 - a)z^2) + \hat{k}((a - 2)x - ax)$

$$= 0 \hat{i} + (3z^2 - a) \hat{j} + (4 - (a + a)) \hat{k}$$
Using (1),
\[ 0 \mathbf{i} + (-4 + a)z^2 \mathbf{j} + (-4 + a)x \mathbf{k} = 0 \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} \]

Comparing the corresponding components both sides,
\[ -4 + a = 0 \quad \Rightarrow \quad a = 4. \]

**Example 34:** If \( \vec{f} = (xy^2) \mathbf{i} + 2x^2yz \mathbf{j} - 3yz^2 \mathbf{k} \), find the \( \text{curl} \ \vec{f} \) at the point (1, -1, 1).

**Solution:** \( \text{curl} \ \vec{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix} = \mathbf{i}(-3z^2 - 2x^2y) + \mathbf{j}(0 - 0) + \mathbf{k}(4xyz - 2xy) \)

At (1, -1, 1),
\[ \text{curl} \ \vec{f} = \mathbf{i}(-3(-1)^2 - 2(1)^2(-1)) + \mathbf{j}(0 - 0) + \mathbf{k}(4(1)(-1)(-1) - 2(1)(-1)) = -\mathbf{i} - 2\mathbf{k}. \]

### 17.7 DEL APPLIED TO THE PRODUCT OF POINT FUNCTIONS

Let \( \phi, \psi \) are two scalar point functions and \( \vec{f}, \vec{g} \) are two vector point functions, then
1. \( \nabla(\phi \psi) = \phi \nabla \psi + \psi \nabla \phi \)
2. \( \nabla \cdot (\phi \vec{f}) = (\nabla \phi) \cdot \vec{f} + \phi (\nabla \cdot \vec{f}) \)
3. \( \nabla \times (\phi \vec{f}) = (\nabla \phi) \times \vec{f} + \phi (\nabla \times \vec{f}) \)
4. \( \nabla(\vec{f} \cdot \vec{g}) = (\vec{f} \cdot \nabla) \vec{g} + (\vec{g} \cdot \nabla) \vec{f} + \vec{f} \times (\nabla \times \vec{g}) + \vec{g} \times (\nabla \times \vec{f}) \) \[\text{[KUK 2007]}\]
5. \( \nabla \cdot (\vec{f} \times \vec{g}) = \vec{g} \cdot (\nabla \times \vec{f}) - \vec{f} \cdot (\nabla \times \vec{g}) \)
6. \( \nabla \times (\vec{f} \times \vec{g}) = \vec{g} \times (\nabla \cdot \vec{f}) - \vec{g} \times (\nabla \times \vec{f}) - \vec{f} \times (\nabla \cdot \vec{g}) + \vec{f} \times (\nabla \times \vec{g}) \) \[\text{[KUK 2011]}\]

**Proof 1:** Consider \( \nabla(\phi \psi) = \sum \mathbf{i} \frac{\partial}{\partial x} (\phi \psi) = \sum \mathbf{i} \left( \phi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x} \right) = \sum \mathbf{i} \left( \phi \frac{\partial \psi}{\partial x} \right) + \sum \mathbf{i} \left( \psi \frac{\partial \phi}{\partial x} \right) = \phi \sum \mathbf{i} \frac{\partial \psi}{\partial x} + \psi \sum \mathbf{i} \frac{\partial \phi}{\partial x} = \phi \nabla \psi + \psi \nabla \phi \)

**Proof 2:** Consider \( \nabla \cdot (\phi \vec{f}) = \sum \mathbf{i} \cdot \frac{\partial}{\partial x} (\phi \vec{f}) = \sum \mathbf{i} \cdot \left( \frac{\partial \phi}{\partial x} \vec{f} + \phi \frac{\partial \vec{f}}{\partial x} \right) = \sum \mathbf{i} \left( \frac{\partial \phi}{\partial x} \right) + \sum \mathbf{i} \left( \phi \frac{\partial \vec{f}}{\partial x} \right) = \left( \sum \mathbf{i} \frac{\partial \phi}{\partial x} \right) \cdot \vec{f} + \phi \left( \sum \mathbf{i} \frac{\partial \vec{f}}{\partial x} \right) = (\nabla \phi) \cdot \vec{f} + \phi (\nabla \cdot \vec{f}) \)

**Proof 3:** Consider \( \nabla \times (\phi \vec{f}) = \sum \mathbf{i} \times \frac{\partial}{\partial x} (\phi \vec{f}) = \sum \mathbf{i} \times \left( \frac{\partial \phi}{\partial x} \vec{f} + \phi \frac{\partial \vec{f}}{\partial x} \right) = \sum \mathbf{i} \times \left( \frac{\partial \phi}{\partial x} \right) + \sum \mathbf{i} \times \left( \phi \frac{\partial \vec{f}}{\partial x} \right) = \left( \sum \mathbf{i} \frac{\partial \phi}{\partial x} \right) \times \vec{f} + \phi \left( \sum \mathbf{i} \times \frac{\partial \vec{f}}{\partial x} \right) = (\nabla \phi) \times \vec{f} + \phi (\nabla \times \vec{f}) \)

**Proof 4:** Consider \( \nabla(\vec{f} \cdot \vec{g}) = \sum \mathbf{i} \frac{\partial}{\partial x} (\vec{f} \cdot \vec{g}) = \sum \mathbf{i} \left( \frac{\partial \vec{f}}{\partial x} \cdot \vec{g} + \vec{f} \cdot \frac{\partial \vec{g}}{\partial x} \right) = \left( \sum \mathbf{i} \frac{\partial \vec{f}}{\partial x} \right) \cdot \vec{g} + \sum \mathbf{i} \left( \vec{f} \cdot \frac{\partial \vec{g}}{\partial x} \right) \)

... (1)
But, \[
\mathbf{\hat{g}} \times \left( \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} \right) = \left( \mathbf{\hat{g}} \cdot \frac{\partial \mathbf{f}}{\partial x} \right) \mathbf{i} - (\mathbf{\hat{g}} \cdot \mathbf{j}) \frac{\partial \mathbf{f}}{\partial x}
\]
or \[
\left( \mathbf{\hat{g}} \cdot \frac{\partial \mathbf{f}}{\partial x} \right) \mathbf{i} = \mathbf{\hat{g}} \times \left( \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} \right) + (\mathbf{\hat{g}} \cdot \mathbf{i}) \frac{\partial \mathbf{f}}{\partial x}
\]

So, \[
\Sigma \left( \mathbf{\hat{g}} \cdot \frac{\partial \mathbf{f}}{\partial x} \right) \mathbf{i} = \mathbf{\hat{g}} \times \Sigma \left( \mathbf{i} \times \frac{\partial \mathbf{g}}{\partial x} \right) + \Sigma (\mathbf{\hat{g}} \cdot \mathbf{i}) \frac{\partial \mathbf{g}}{\partial x} = \mathbf{\hat{g}} \times (\nabla \times \mathbf{f}) + (\mathbf{\hat{g}} \cdot \nabla) \mathbf{f}
\] ... (2)

Interchanging \( \mathbf{\hat{f}} \) and \( \mathbf{\hat{g}} \) in (2)

\[
\Sigma (\mathbf{\hat{f}} \cdot \frac{\partial \mathbf{g}}{\partial x}) \mathbf{i} = \mathbf{\hat{f}} \times \Sigma \left( \mathbf{i} \times \frac{\partial \mathbf{g}}{\partial x} \right) + \Sigma (\mathbf{\hat{f}} \cdot \mathbf{i}) \frac{\partial \mathbf{g}}{\partial x} = \mathbf{\hat{f}} \times (\nabla \times \mathbf{g}) + (\mathbf{\hat{f}} \cdot \nabla) \mathbf{g}
\] ... (3)

Using (2) and (3) in (1), we get

\[
\nabla (\mathbf{\hat{f}} \cdot \mathbf{\hat{g}}) = (\mathbf{\hat{f}} \cdot \nabla) \mathbf{g} + (\mathbf{\hat{g}} \cdot \nabla) \mathbf{f} + \mathbf{\hat{f}} \times (\nabla \times \mathbf{g}) + \mathbf{\hat{g}} \times (\nabla \times \mathbf{f})
\]

Proof 5: Consider \( \nabla \cdot (\mathbf{\hat{f}} \times \mathbf{\hat{g}}) = \mathbf{\hat{f}} \cdot (\nabla \times \mathbf{\hat{g}}) = \mathbf{\hat{f}} \times (\nabla \times \mathbf{\hat{g}}) = \mathbf{\hat{f}} \times (\nabla \times \mathbf{g}) + (\mathbf{\hat{f}} \cdot \nabla) \mathbf{g}
\]

Proof 6: Consider \( \nabla \times (\mathbf{\hat{f}} \times \mathbf{\hat{g}}) = \mathbf{\hat{f}} \times (\nabla \times \mathbf{\hat{g}}) = \mathbf{\hat{f}} \times (\nabla \times \mathbf{g}) + (\mathbf{\hat{f}} \cdot \nabla) \mathbf{g}
\]

17.8 DEL APPLIED TWICE TO POINT FUNCTIONS

Let \( \varphi \) be a scalar point function and \( \mathbf{\hat{f}} \) be a vector point function, then \( \nabla \varphi \) and \( \nabla \times \mathbf{\hat{f}} \) being the vector point functions, we can find their divergence and curl; whereas \( \nabla \cdot \mathbf{\hat{f}} \) being the scalar point function, we can find its gradient only. Thus we have following formulae:

1. \( \text{div grad } \varphi = \nabla \cdot (\nabla \varphi) = \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \)
2. \( \text{curl grad } \varphi = \nabla \times \mathbf{\hat{g}} = \mathbf{\hat{0}} \)
3. \( \text{div curl } \mathbf{\hat{f}} = \nabla \cdot (\nabla \times \mathbf{\hat{f}}) = 0 \)
4. \( \text{curl curl } \mathbf{\hat{f}} = \nabla \times (\nabla \times \mathbf{\hat{f}}) = (\nabla \cdot \mathbf{\hat{f}}) - \nabla^2 \mathbf{\hat{f}} = \text{grad div } \mathbf{\hat{f}} - \nabla^2 \mathbf{\hat{f}} \) \[\text{KUK 2006]}\]
5. \( \text{grad div } \mathbf{\hat{f}} = \nabla (\mathbf{\hat{f}} \cdot \nabla) = \text{curl curl } \mathbf{\hat{f}} + \nabla^2 \mathbf{\hat{f}} = \nabla \times (\nabla \times \mathbf{\hat{f}}) + \nabla^2 \mathbf{\hat{f}} \)
Proof 1: \( \nabla^2 \phi = \nabla \cdot (\nabla \phi) = \nabla \cdot \left( \left( i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) \right) = \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \phi}{\partial z} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \)

Here \( \nabla^2 \) is called the Laplacian Operator and \( \nabla^2 \phi = 0 \) is called the Laplace’s Equation.

Proof 2: \( \text{curl} \, \text{grad} \, \phi = \nabla \times \nabla \phi = \nabla \times \left( \left( i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) \right) = \sum \vec{i} \left( \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) = 0 \)

Proof 3: \( \text{div} \, \text{curl} \, \vec{f} = \nabla \cdot (\nabla \times \vec{f}) = \left( \sum i \frac{\partial}{\partial x} \right) \times \left( \left( i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \right) \right) = \sum i \left( i \frac{\partial^2 f}{\partial x^2} + j \frac{\partial^2 f}{\partial x \partial y} + k \frac{\partial^2 f}{\partial x \partial z} \right) \)

Proof 4: \( \text{curl} \, \text{curl} \, \vec{f} = \nabla \times (\nabla \times \vec{f}) = \left( \sum i \frac{\partial}{\partial x} \right) \times \left( \left( i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \right) \right) = \sum i \left( i \frac{\partial^2 f}{\partial x^2} + j \frac{\partial^2 f}{\partial x \partial y} + k \frac{\partial^2 f}{\partial x \partial z} \right) \)

Proof 5: To get this formula we are to re-arrange the terms in last proof.

Example 35: Show that \( \nabla^2 (r^n) = n(n+1)r^{n-2} \).

Solution: We know that \( r^2 = x^2 + y^2 + z^2 \)

On differentiation w. r. t. x, \( \frac{\partial r}{\partial x} = \frac{x}{r} \)

Similarly \( \frac{\partial r}{\partial y} = \frac{y}{r} \) and \( \frac{\partial r}{\partial z} = \frac{z}{r} \)

Now \( \nabla^2 (r^n) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (r^n) = \frac{\partial}{\partial x} \left( \frac{\partial r^n}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial r^n}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial r^n}{\partial z} \right) \)

And \( \frac{\partial}{\partial x} \left( \frac{\partial r^n}{\partial x} \right) = \frac{\partial}{\partial x} \left( n r^{n-1} \frac{\partial r}{\partial x} \right) = n \frac{\partial}{\partial x} \left( r^{n-1} \frac{x}{r} \right) = n \frac{\partial}{\partial x} \left( r^{n-2} x \right) \)
\[ n \left( r^{n-2} + x (n-2) r^{n-3} \frac{\partial r}{\partial x} \right) = n (r^{n-2} + x^2 (n-2) r^{n-4}) \]

Similarly
\[ \frac{\partial}{\partial y} \left( \frac{\partial r^n}{\partial y} \right) = n (r^{n-2} + y^2 (n-2) r^{n-4}) \]
\[ \frac{\partial}{\partial z} \left( \frac{\partial r^n}{\partial z} \right) = n (r^{n-2} + z^2 (n-2) r^{n-4}) \]

Using all these values in (1),
\[ \nabla^2 (r^n) = n(r^{n-2} + x^2 (n-2) r^{n-4}) + n(r^{n-2} + (n-2) r^{n-4}) + n(r^{n-2} + z^2 (n-2) r^{n-4}) \]
\[ = 3n r^{n-2} + n(n-2) r^{n-4} (x^2 + y^2 + z^2) \]
\[ = 3n r^{n-2} + n(n-2) r^{n-2} = n(n+1)r^{n-2} \]

**Example 36:** Show that *(i) \( \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r) \) (ii) \( \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi \)

* [KUK 2008]

**Solution:** *(i) \( \nabla^2 f(r) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f(r) = \frac{\partial}{\partial x} \left( \frac{\partial f(r)}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f(r)}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial f(r)}{\partial z} \right) \ldots (1) \)

And
\[ \frac{\partial}{\partial x} \left( \frac{\partial f(r)}{\partial x} \right) = \frac{\partial}{\partial x} \left( f'(r) \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} \left( f'(r) \frac{\partial}{\partial x} \left( f\left( \frac{x}{r} \right) \right) \right) = f'(r) \frac{\partial}{\partial x} \left( f\left( \frac{x}{r} \right) \right) + \frac{x}{r} f''(r) \frac{\partial x}{\partial x} \]
\[ = f'(r) \left( 1 - \frac{x}{r^2} \frac{\partial r}{\partial x} \right) + \frac{x}{r} f''(r) \frac{\partial x}{\partial x} \]
\[ = f'(r) - x^2 \frac{f'(r)}{r^3} + \frac{x}{r} f''(r) \]

Similarly
\[ \frac{\partial}{\partial y} \left( \frac{\partial f(r)}{\partial y} \right) = \frac{f'(r)}{r} - y^2 \frac{f'(r)}{r^3} + y^2 f''(r) \frac{r}{r^2} \]
\[ \frac{\partial}{\partial z} \left( \frac{\partial f(r)}{\partial z} \right) = \frac{f'(r)}{r} - z^2 \frac{f'(r)}{r^3} + z^2 f''(r) \frac{r}{r^2} \]

Using all these values in (1)
\[ \nabla^2 f(r) = \left( \frac{f'(r)}{r} - x^2 \frac{f'(r)}{r^3} + \frac{x}{r} f''(r) \right) + \left( \frac{f'(r)}{r} - y^2 \frac{f'(r)}{r^3} + y^2 f''(r) \frac{r}{r^2} \right) \]
\[ + \left( \frac{f'(r)}{r} - z^2 \frac{f'(r)}{r^3} + z^2 f''(r) \frac{r}{r^2} \right) \]
\[ = \frac{3f'(r)}{r} - (x^2 + y^2 + z^2) \frac{f'(r)}{r^3} + f''(r) (x^2 + y^2 + z^2) \]
\[ = \frac{3f'(r)}{r} - \frac{f'(r)}{r^3} + f''(r) \]
\[ = 2 \frac{f'(r)}{r} + f''(r) \]

Hence \( \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r) \).

(ii) Consider \( \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \nabla \cdot (\phi \nabla \psi) - \nabla \cdot (\psi \nabla \phi) \)
\[ = \{ \nabla \phi \cdot \nabla \psi + \phi (\nabla \cdot \nabla \psi) \} - \{ \nabla \psi \cdot \nabla \phi + \psi (\nabla \cdot \nabla \phi) \} \]
\[ = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi - \nabla \phi \cdot \nabla \psi - \psi \nabla^2 \phi \]
\[ = \phi \nabla^2 \psi - \psi \nabla^2 \phi \]

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Hence \( \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi. \)

Example 37: Find the value of \( n \) for which the vector \( r^n \vec{r} \) is solenoidal, where \( \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \).

**Solution:** Consider \( \text{div}(r^n \vec{r}) = \nabla \cdot (r^n \vec{r}) = (\nabla r^n) \cdot \vec{r} + r^n (\nabla \cdot \vec{r}) \) \( \ldots (1) \)

But \[ \nabla r^n = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) r^n = n r^{n-1} \left( \frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z} \right) \]

\[ = n r^{n-1} \left( \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j} + \frac{z}{r} \hat{k} \right) = n r^{n-2} \vec{r} \] \( \ldots (2) \)

And \( \nabla \cdot \vec{r} = 3 \) \( \ldots (3) \)

Therefore, \( \text{div}(r^n \vec{r}) = (n r^{n-2} \vec{r}) \cdot \vec{r} + (3) r^n = n r^{n-2} (\vec{r} \cdot \vec{r}) + 3 r^n \)

\[ = n r^{n-2} (r^2) + 3 r^n = (n + 3) r^n \] \( \ldots (4) \)

As given the vector \( r^n \vec{r} \) is solenoidal, so \( \text{div}(r^n \vec{r}) = 0 \)

So using (4), \( (n + 3) r^n = 0 \) implies that \( n = -3 \) (since \( r \neq 0 \))

Example 38: If \( \vec{a} \) and \( \vec{b} \) are irrotational, prove that \( \vec{a} \times \vec{b} \) is solenoidal.

**Solution:** Given \( \vec{a} \) and \( \vec{b} \) are irrotational, so \( \nabla \times \vec{a} = \vec{0} = \nabla \times \vec{b} \) \( \ldots (1) \)

Consider \( \text{Div} \left( \vec{a} \times \vec{b} \right) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}) = \vec{b} \cdot \vec{0} - \vec{a} \cdot \vec{0} = 0 \) [using (1)]

Thus \( \vec{a} \times \vec{b} \) is solenoidal.

Example 39: Show that the vector field \( \vec{f} = (z^2 + 2x + 3y)\hat{i} + (3x + 2y + z)\hat{j} + (y + 2zx)\hat{k} \) is irrotational but not solenoidal. Also obtain a scalar function \( \phi \) such that \( \nabla \phi = \vec{f} \).

**Solution:** Consider \( \text{Curl} \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 + 2x + 3y & 3x + 2y + z & y + 2zx \end{vmatrix} \)

\[ = \hat{i}(1 - 1) - \hat{j}(2z - 2z) + \hat{k}(3 - 3) = \vec{0} \]

So \( \vec{f} \) is irrotational vector field.

Also consider \( \text{div} \vec{f} = \nabla \cdot \vec{f} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot \vec{f} = 2 + 2 + 2x = 2(x + 2) \neq 0 \)

So \( \vec{f} \) is not solenoidal vector field.

Now \[ d\phi = \frac{\partial \phi}{\partial x} \, dx + \frac{\partial \phi}{\partial y} \, dy + \frac{\partial \phi}{\partial z} \, dz = \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} \right) \cdot (dx \, \hat{i} + dy \, \hat{j} + dz \, \hat{k}) \]

\[ = \text{grad} \, \phi \cdot d\vec{r} = \vec{f} \cdot d\vec{r} \] (as \( \vec{f} = \text{grad} \, \phi \))

\[ = \left( (z^2 + 2x + 3y)\hat{i} + (3x + 2y + z)\hat{j} + (y + 2zx)\hat{k} \right) \cdot (dx \, \hat{i} + dy \, \hat{j} + dz \, \hat{k}) \]

\[ = (z^2 + 2x + 3y)dx + (3x + 2y + z)dy + (y + 2zx)dz \]

\[ = (z^2dx + 2zdx) + (3y dx + 3x dy) + (z dy + y dz) + 2x dx + 2y dy \]

\[ = d(xz^2) + 3d(xy) + d(yz) + d(x^2) + d(y^2) \]

Integrating both sides, we get
\[ \emptyset = xz^2 + 3xy + yz + x^2 + y^2 + c \]

Example 40: If \( \vec{V}_1 \) and \( \vec{V}_2 \) be the vectors joining the fixed points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) respectively to a variable point \((x, y, z)\), prove that *\([KUK 2010]\)

(i) \( \text{div} (\vec{V}_1 \times \vec{V}_2) = 0 \)  *(ii) \( \text{curl} (\vec{V}_1 \times \vec{V}_2) = 2(\vec{V}_1 - \vec{V}_2) \)  (iii) \( \text{grad} (\vec{V}_1 \cdot \vec{V}_2) = \vec{V}_1 + \vec{V}_2. \)

**Solution:** Here, \( \vec{V}_1 = (x-x_1)i + (y-y_1)j + (z-z_1)k \) and \( \vec{V}_2 = (x-x_2)i + (y-y_2)j + (z-z_2)k \)

(i) \( \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} i & j & k \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix} \)

\[ = [(y-y_1)(z-z_2) - (y-y_2)(z-z_1)]i + [(x-x_2)(z-z_1) - (x-x_1)(z-z_2)]j \]
\[ + [(x-x_1)(y-y_2) - (x-x_2)(y-y_1)]k \]

So \( \text{div} (\vec{V}_1 \times \vec{V}_2) = \frac{\partial}{\partial x} [(y-y_1)(z-z_2) - (y-y_2)(z-z_1)] \]
\[ + \frac{\partial}{\partial y} [(x-x_2)(z-z_1) - (x-x_1)(z-z_2)] \]
\[ + \frac{\partial}{\partial z} [(x-x_1)(y-y_2) - (x-x_2)(y-y_1)] = 0 \]

(ii) \( \text{curl} (\vec{V}_1 \times \vec{V}_2) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y-y_1)(z-z_2) - (y-y_2)(z-z_1) & (x-x_2)(z-z_1) - (x-x_1)(z-z_2) & (x-x_1)(y-y_2) - (x-x_2)(y-y_1) \end{vmatrix} \)

\[ = [(x-x_1) - (x-x_2) - (x-x_2) + (x-x_1)]i \]
\[ + [(y-y_1) - (y-y_2) - (y-y_2) + (y-y_1)]j \]
\[ + [(z-z_1) - (z-z_2) - (z-z_2) + (z-z_1)]k \]
\[ = 2[(x-x_1)i + (y-y_1)j + (z-z_1)k] - 2[(x-x_2)i + (y-y_2)j + (z-z_2)k] \]
\[ = 2(\vec{V}_1 - \vec{V}_2) \]

(iii) \( \vec{V}_1 \cdot \vec{V}_2 = (x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) \)

So \( \text{grad} (\vec{V}_1 \cdot \vec{V}_2) = \frac{\partial}{\partial x} [(x-x_1)(x-x_2)] + \frac{\partial}{\partial y} [(y-y_1)(y-y_2)] + \frac{\partial}{\partial z} [(z-z_1)(z-z_2)] \)
\[ = i[(x-x_1) + (x-x_2)] + j[(y-y_1) + (y-y_2)] + k[(z-z_1) + (z-z_2)] \]
\[ = [(x-x_1)i + (y-y_1)j + (z-z_1)k] + [(x-x_2)i + (y-y_2)j + (z-z_2)k] \]
\[ = \vec{V}_1 + \vec{V}_2 \]

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Example 41: If \( \vec{a} \) is a constant vector and \( \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \), prove that \( \begin{align*}
(i) \quad \text{grad} (\vec{a} \cdot \vec{r}) &= \vec{a} \\
(ii) \quad \text{div} (\vec{a} \times \vec{r}) &= 0 \\
(iii) \quad \text{curl} (\vec{a} \times \vec{r}) &= 2\vec{a} \\
(iv) \quad \text{curl} [(\vec{a} \cdot \vec{r})\vec{r}] &= \vec{a} \times \vec{r}
\end{align*} \) 

Solution: Let \( \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \) is the constant vector.

(i) \( \vec{a} \cdot \vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = (a_1x + a_2y + a_3z) \)

So \( \text{grad} (\vec{a} \cdot \vec{r}) = \frac{\partial}{\partial x} [a_1x + a_2y + a_3z] + \frac{\partial}{\partial y} [a_1x + a_2y + a_3z] + \frac{\partial}{\partial z} [a_1x + a_2y + a_3z] \)

\[= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a} \]

(ii) \( \vec{a} \times \vec{r} = \begin{vmatrix} i & j & k \\
a_1 & a_2 & a_3 \\
x & y & z \end{vmatrix} = i(a_2z - a_3y) + j(a_3x - a_1z) + k(a_1y - a_2x) \)

\( \text{div} (\vec{a} \times \vec{r}) = \frac{\partial}{\partial x} (a_2z - a_3y) + \frac{\partial}{\partial y} (a_3x - a_1z) + \frac{\partial}{\partial z} (a_1y - a_2x) = 0 \)

(iii) \( \text{curl} (\vec{a} \times \vec{r}) = \begin{vmatrix} i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
(a_2z - a_3y) & (a_3x - a_1z) & (a_1y - a_2x) \end{vmatrix} = i(a_1 + a_1) + j(a_2 + a_2) + k(a_3 + a_3) = 2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 2\vec{a} \)

(iv) \( (\vec{a} \cdot \vec{r})\vec{r} = (a_1x + a_2y + a_3z)x\hat{i} + (a_1x + a_2y + a_3z)y\hat{j} + (a_1x + a_2y + a_3z)z\hat{k} \)

\( \text{curl} [(\vec{a} \cdot \vec{r})\vec{r}] = \begin{vmatrix} i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
(a_1x + a_2y + a_3z)x & (a_1x + a_2y + a_3z)y & (a_1x + a_2y + a_3z)z \end{vmatrix} = i(a_2z - a_3y) + j(a_3x - a_1z) + k(a_1y - a_2x) = \vec{a} \times \vec{r} \) [using part (ii)]

Example 42: Find \( \vec{f} \times (\nabla \times \vec{g}) \) at the point (1, -1, 2),

if \( \vec{f} = xz^2\hat{i} + 2y\hat{j} - 3xz\hat{k}, \vec{g} = 3xz\hat{i} + 2yz\hat{j} - z^2\hat{k} \).

Solution: \( \nabla \times \vec{g} = \begin{vmatrix} i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
3xz & 2yz & -z^2 \end{vmatrix} = i(0 - 2y) + j(3x - 0) + k(0 - 0) = -2y\hat{i} + 3x\hat{j} + 0\hat{k} \)

Now \( \vec{f} \times (\nabla \times \vec{g}) = \begin{vmatrix} i & j & k \\
xz^2 & 2y & -3xz \end{vmatrix} = (9x^2z)\hat{i} + (6xyz)\hat{j} + (3x^2z^2 + 4y^2)\hat{k} \)

At (1, -1, 2), \( \vec{f} \times (\nabla \times \vec{g}) = (9(1)^2(2))\hat{i} + (6(1)(-1)(2))\hat{j} + (3(1)^2(2)^2 + 4(-1)^2)\hat{k} \)

\[= 18\hat{i} - 12\hat{j} + 16\hat{k} \]

Example 43: If \( \vec{f} = yz^2\hat{i} - 3xz^2\hat{j} + 2xyz\hat{k} \), \( \vec{g} = 3x\hat{i} + 4z\hat{j} - xy\hat{k} \) and \( \phi = xyz \); find

(i) \( \vec{f} \times \nabla \phi \) \quad (ii) \( (\vec{f} \times \nabla)\phi \) \quad (iii) \( (\nabla \times \vec{f}) \times \vec{g} \) \quad (iv) \( \vec{g} \cdot (\nabla \times \vec{f}) \)

Solution: \( \nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} = yz\hat{i} + zx\hat{j} + xy\hat{k} \)

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(i) \( \mathbf{f} \times \nabla = \begin{vmatrix} i & j & k \\ yz^2 & -3xz^2 & 2xyz \\ yz & xz & xy \end{vmatrix} = -5x^2yz \ \hat{i} + xy^2z^2 \ \hat{j} + 4xyz^3 \ \hat{k} \)

(ii) \( \nabla \times \mathbf{f} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = i \{ -3xz^2 \frac{\partial}{\partial z} - 2xyz \frac{\partial}{\partial y} \} + j \{ 2xyz \frac{\partial}{\partial x} - yz^2 \frac{\partial}{\partial y} \} + k \{ yz^2 \frac{\partial}{\partial y} + 3xz^2 \frac{\partial}{\partial x} \} \)

Now \( \nabla \times \mathbf{f} = i \{ -3xz^2 (y) - 2xyz (x) \} + j \{ 2xyz (y) - yz^2 (x) \} + k \{ yz^2 (x) + 3xz^2 (y) \} \)

\( = -5x^2yz \ \hat{i} + xy^2z^2 \hat{j} + 4xyz^3 \ \hat{k} \)

(iii) \( \mathbf{\nabla} \times \mathbf{f} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8xz & 0 & -4z^2 \\ xz & 4z & -xy \end{vmatrix} = (0 + 16z^3) \ \hat{i} + (-12xz^2 + 8x^2yz) \ \hat{j} + (32xz^2 - 0) \ \hat{k} \)

\( = 16z^3 \ \hat{i} + (-12xz^2 + 8x^2yz) \ \hat{j} + 32xz^2 \ \hat{k} \)

\( \mathbf{g} \cdot (\mathbf{\nabla} \times \mathbf{f}) = (3x \ \hat{i} + 4z \ \hat{j} - xy \ \hat{k}) \cdot (8xz \ \hat{i} + 0 \ \hat{j} - 4z^2 \ \hat{k}) = 24x^2z + 4xyz^2 \)

Example 44: Find the directional derivative of \( \mathbf{\nabla} \cdot (\mathbf{\nabla} \cdot \mathbf{f}) \) at the point \( (1, -2, 1) \) in the direction of the normal to the surface \( xy^2z = 3x + z^2 \) where \( \mathbf{f} = 2x^3y^2z^4 \).

Solution: Given \( \mathbf{f} = 2x^3y^2z^4 \)

So \( \mathbf{\nabla} \cdot \mathbf{f} = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} = \hat{i} (6x^2y^2z^4) + \hat{j} (4x^3yz^4) + \hat{k} (8x^3y^2z^3) \)

And \( \mathbf{\nabla} \cdot (\mathbf{\nabla} \cdot \mathbf{f}) = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \)

Consider \( \text{grad}(f) = \mathbf{\nabla} f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \)

\( = \hat{i} (12y^2z^4 + 12x^2z^4 + 72x^3y^2z^2) + \hat{j} (24xyz^4 + 48x^3yz^2) + \hat{k} (48x^2y^2z^3 + 16x^3z^3 + 48x^3y^2z) \)

At point \( (1, -2, 1) \)

\( \text{grad}(f) = \hat{i} (48 + 12 + 288) + \hat{j} (-48 - 96) + \hat{k} (192 + 16 + 192) \)

\( = 348 \ \hat{i} - 144 \ \hat{j} + 400 \ \hat{k} \)

Now consider the surface \( g = xy^2z - 3x - z^2 = 0 \)
So, \( \nabla g = \hat{i} \frac{\partial g}{\partial x} + \hat{j} \frac{\partial g}{\partial y} + \hat{k} \frac{\partial g}{\partial z} = \hat{i} (y^2 z - 3) + \hat{j} (2xyz) + \hat{k} (xy^2 - 2z) \)

The vector normal to the surface (1) at point (1, -2, 1) is given by

\[ \vec{n} = \hat{i} - 4\hat{j} + 2\hat{k} \]

And \( |\vec{n}| = \sqrt{1 + 16 + 4} = \sqrt{21} \)

So the direction derivative of \( f = \nabla \cdot (\nabla \phi) \) at the point (1, -2, 1) in the direction of the normal to the surface (1) is \( \text{grad}(f) \cdot \frac{\vec{n}}{|\vec{n}|} \)

\[ = \left(348 \hat{i} - 144 \hat{j} + 400 \hat{k}\right) \cdot \frac{1}{\sqrt{21}} \left(\hat{i} - 4\hat{j} + 2\hat{k}\right) \]

\[ = \frac{1}{\sqrt{21}} (348 + 576 + 800) \]

\[ = 1724/\sqrt{21} \]

**Example 45:** If \( r \) is the distance of a point \((x, y, z)\) from the origin, prove that \( \text{curl} \left( \hat{k} \times \text{grad} \left( \frac{1}{r} \right) \right) + \text{grad} \left( \hat{k} \cdot \text{grad} \left( \frac{1}{r} \right) \right) = \vec{0} \) where \( \hat{k} \) is the unit vector in the direction of OZ.

**Solution:** Here \( \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \) so that \( r = \sqrt{x^2 + y^2 + z^2} \)

Now \( \text{grad} \left( \frac{1}{r} \right) = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \frac{1}{r} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{-\frac{1}{2}} \)

\[ = -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} (2x \hat{i} + 2y \hat{j} + 2z \hat{k}) = -\frac{1}{r^3} \vec{r} \]

\[ \text{grad} \left( \hat{k} \cdot \text{grad} \left( \frac{1}{r} \right) \right) = \text{grad} \left( -\frac{1}{r^3} z \right) = -\text{grad} \left( \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \]

\[ = \frac{3zx}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \hat{i} + \frac{3zy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \hat{j} + \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \hat{k} \]

(1)

And \( \text{curl} \left( \hat{k} \times \text{grad} \left( \frac{1}{r} \right) \right) = \text{curl} \left( \hat{k} \times -\frac{1}{r^3} \vec{r} \right) = \text{curl} \left( \frac{1}{r^3} (y \hat{i} - x \hat{j}) \right) \]

\[ = -\frac{3zx}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \hat{i} - \frac{3zy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \hat{j} - \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \hat{k} \]

(2)

Adding (1) and (2), \( \text{curl} \left( \hat{k} \times \text{grad} \left( \frac{1}{r} \right) \right) + \text{grad} \left( \hat{k} \cdot \text{grad} \left( \frac{1}{r} \right) \right) = \vec{0} \).

**Example 46:** Prove that \( \vec{a} \cdot \nabla \left( \vec{b} \cdot \nabla \left( \frac{1}{r} \right) \right) = \frac{3(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r})}{r^5} - \frac{\vec{a} \vec{b}}{r^3} \), where \( \vec{a} \) and \( \vec{b} \) are constant vectors.

**Solution:** From example 45, \( \nabla \left( \frac{1}{r} \right) = -\frac{1}{r^3} \vec{r} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} (x \hat{i} + y \hat{j} + z \hat{k}) \)

Let the constant vectors are \( \vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \) and \( \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} \)

So \( \nabla \left( \vec{b} \cdot \nabla \left( \frac{1}{r} \right) \right) = \nabla \left( -(x^2 + y^2 + z^2)^{-\frac{3}{2}} (b_1 x + b_2 y + b_3 z) \right) \)

\[ = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} \nabla (b_1 x + b_2 y + b_3 z) - (b_1 x + b_2 y + b_3 z) \nabla (x^2 + y^2 + z^2)^{-\frac{3}{2}} \]

\[ = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) + (b_1 x + b_2 y + b_3 z) (x^2 + y^2 + z^2)^{-\frac{5}{2}} (x \hat{i} + y \hat{j} + z \hat{k}) \]

\[ = \frac{\vec{b}}{r^3} + \frac{3(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r})}{r^5} \]
Now \( \vec{a} \cdot \nabla \left( \frac{\vec{b} \cdot \nabla \vec{a}}{r} \right) = \vec{a} \cdot \left( -\frac{\vec{b}}{r^3} + \frac{3(\vec{b} \cdot \nabla)\vec{a}}{r^5} \right) = \frac{3(\vec{a} \cdot \nabla)(\vec{b} \cdot \nabla) \vec{a}}{r^5} - \frac{\vec{a} \cdot \vec{b}}{r^3} \)

Hence Proved.

**ASSIGNMENT 4**

1. If \( \vec{f} = (x + y + 1)\hat{i} + f - (x + y)\hat{k} \), show that \( \vec{f} \cdot \text{curl} \vec{f} = 0 \).
2. Evaluate (a) \( \text{div} \left[ 3x^2\hat{i} + 5xy^2\hat{j} + xyz^2\hat{k} \right] \) at the point (1, 2, 3).
   (b) \( \text{curl} \left[ e^{xyz} (\hat{i} + f + \hat{k}) \right] \).
3. Find the value of ‘a’ if the vector \((ax^2y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)\hat{k}\) has zero divergence. Find the curl of above vector which has zero divergence.
4. If \( \vec{v} = \frac{\vec{r}}{\sqrt{x^2 + y^2 + z^2}} \), show that \( \nabla \cdot \vec{v} = 2/\sqrt{x^2 + y^2 + z^2} \) and \( \nabla \times \vec{v} = \vec{0} \).
5. If \( u = x^2 + y^2 + z^2 \) and \( \vec{v} = x\hat{i} + y\hat{j} + zk \), show that \( \text{div} \ (u \vec{v}) = 5u \).
6. Show that each of following vectors are solenoidal:
   (a) \( (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k} \) (b) \( 3y^4z^2\hat{i} + 4x^3z^2\hat{j} + 3x^2y^2\hat{k} \) (c) \( \text{rot} \ (\nabla \times \vec{v}) \).
7. If \( \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \) and \( r = |\vec{r}| \neq 0 \), show that
   (a) \( \nabla (1/r^2) = -2\vec{r}/r^4 \), \( \nabla \cdot (\vec{r}^2/r^2) = 1/r^2 \)
   (b) \( \nabla \cdot (r^n\vec{r}) = (n + 3)r^n \), \( \nabla \times (r^n\vec{r}) = \vec{0} \).
8. Prove that (a) \( \nabla \vec{a}^2 = 2(\vec{a} \cdot \nabla)\vec{a} + 2\vec{a} \times (\nabla \times \vec{a}) \), where \( \vec{a} \) is the constant vector.

(b) \( \nabla \times (\vec{r} \times \vec{u}) = \vec{r}(\nabla \cdot \vec{u}) - 2\vec{u} - (\vec{r} \cdot \nabla)\vec{u} \).
9. (a) If \( \phi = (x^2 + y^2 + z^2)^{-n} \), find the \( \text{div} \text{ grad} \phi \) and determine \( n \) if \( \text{div} \text{ grad} \phi = 0 \).
   (b) Show that \( \text{grad} r^n = n(n + 1)r^{n-2} \), where \( r^2 = x^2 + y^2 + z^2 \).
10. In electromagnetic theory, we have \( \nabla \cdot \vec{B} = \rho, \ \nabla \cdot \vec{H} = 0, \ \nabla \times \vec{B} = -\frac{1}{c} \frac{\partial \vec{E}}{\partial t}, \ \nabla \times \vec{H} = \frac{1}{c} \left( \rho \vec{V} + \frac{\partial \vec{B}}{\partial t} \right) \).

Prove that \( \nabla^2 \vec{H} - \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2} = -\frac{1}{c} \nabla \times (\rho \vec{V}) \) and \( \nabla^2 \vec{D} - \frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2} = \nabla \rho + \frac{1}{c^2} \frac{\partial}{\partial t} (\vec{P} \vec{V}) \).
11. If \( u = x^2yz, \ v = xy - 3z^2 \), find (i) \( \nabla(\nabla u \times \nabla v) \) (ii) \( \nabla \cdot (\nabla u \times \nabla v) \).
12. For a solenoidal vector \( \vec{f} \), show that \( \text{curl} \text{ curl} \text{ curl} \vec{f} = \nabla^4 \vec{f} \).
13. Calculate (i) \( \text{curl} (\text{grad} f) \), given \( f(x, y, z) = x^2 + y^2 - z \) \[\text{BPTU 2006}\]
   (ii) \( \text{curl} (\text{curl} \vec{a}) \), given \( \vec{a} = x^2y\hat{i} + y^2z\hat{j} + z^2y\hat{k} \)
14. Show that each of the following vectors are solenoidal:
   (i) \( -(x^2 + yz)\hat{i} + (4y - z^2x)\hat{j} + (2xz - 4z)\hat{k} \)
   (ii) \( 3y^4z^2\hat{i} + 4x^3z^2\hat{j} + 3x^2y^2\hat{k} \)
   (iii) \( \nabla \phi \times \nabla \psi \)

**INTEGRAL VECTOR CALCULUS**

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17.9 INTEGRATION OF VECTORS

If two vector functions \( \vec{f}(t) \) and \( \vec{g}(t) \) be such that \( \frac{d}{dt}(\vec{g}(t)) = \vec{f}(t) \), then \( \vec{g}(t) \) is called an integral of \( \vec{f}(t) \) with respect to a scalar variable \( t \) and we can write \( \int \vec{f}(t)\,dt = \vec{g}(t) \).

Indefinite Integral

If \( \vec{c} \) be an arbitrary constant vector and \( \frac{d}{dt}(\vec{g}(t)) = \frac{d}{dt}(\vec{f}(t) + \vec{c}) \), then \( \int \vec{f}(t)\,dt = \vec{g}(t) + \vec{c} \). This is called the indefinite integral of \( \vec{f}(t) \).

Definite Integral

If \( \frac{d}{dt}(\vec{g}(t)) = \frac{d}{dt}(\vec{f}(t)) \) for all values of \( t \) in the interval \( (a, b) \), then the definite integral of \( \vec{f}(t) \) between \( a \) and \( b \) is defined and denoted by \( \int_a^b \vec{f}(t)\,dt = [\vec{g}(t)]_a^b = \vec{g}(b) - \vec{g}(a) \).

Example 47: If \( \frac{d^2\vec{f}}{dt^2} = 6t \hat{i} - 12t^2 \hat{j} + 4 \cos t \ \hat{k} \), find \( \vec{f} \), given that \( \frac{d\vec{f}}{dt} = -\hat{i} - 3 \hat{k} \) and \( \vec{f} = 2 \hat{i} + \hat{j} \) at \( t = 0 \).

Solution: Given that \( \frac{d^2\vec{f}}{dt^2} = 6t \hat{i} - 12t^2 \hat{j} + 4 \cos t \ \hat{k} \) \( \ldots \) (1)

Integrating (1) with respect to \( t \),

\[ \int \frac{d^2\vec{f}}{dt^2} \,dt = \int (6t \hat{i} - 12t^2 \hat{j} + 4 \cos t \ \hat{k}) \,dt \]

Implies

\[ \frac{d\vec{f}}{dt} = 3t^2 \hat{i} - 4t^3 \hat{j} + 4 \sin t \ \hat{k} + \vec{c}_1 \] \( \ldots \) (2)

Now, integrating (2) with respect to \( t \),

\[ \int \frac{d\vec{f}}{dt} \,dt = \int (3t^2 \hat{i} - 4t^3 \hat{j} + 4 \sin t \ \hat{k} + \vec{c}_1) \,dt \]

Implies

\[ \vec{f} = t^3 \hat{i} - t^4 \hat{j} - 4 \cos t \ \hat{k} + \vec{c}_1 t + \vec{c}_2 \] \( \ldots \) (3)

Also we are given that at \( t = 0 \), \( \frac{d\vec{f}}{dt} = -\hat{i} - 3 \hat{k} \) \( \ldots \) (4)

\[ \vec{f} = 2 \hat{i} + \hat{j} \] \( \ldots \) (5)

Using (2) and (4), \( \vec{c}_1 = -\hat{i} - 3 \hat{k} \)

Using (3) and (5), \(-4 \ \hat{k} + \vec{c}_2 = 2 \hat{i} + \hat{j} \) \( \Rightarrow \vec{c}_2 = 2 \hat{i} + \hat{j} + 4 \ \hat{k} \)

Putting the values of the constant vectors \( \vec{c}_1 & \vec{c}_2 \) in (3), we get

\[ \vec{f} = t^3 \hat{i} - t^4 \hat{j} - 4 \cos t \ \hat{k} + (-\hat{i} - 3 \hat{k}) t + (2 \hat{i} + \hat{j} + 4 \ \hat{k}) \]

\[ = (t^3 - t^2 + 2) \hat{i} + (1 - t^4) \hat{j} - (4 \cos t + 3 t + 4) \ \hat{k} \]

ASSIGNMENT 5

1. For given \( \vec{f}(t) = (5t^2 - 3t) \hat{i} + 6t^3 \hat{j} - 7t \ \hat{k} \), evaluate \( \int_2^4 \vec{f}(t)\,dt \).

2. Given \( \vec{r}(t) = 3t^2 \hat{i} + t \hat{j} - t^3 \ \hat{k} \), evaluate \( \int_0^1 (\vec{r} \times \frac{d^2\vec{r}}{dt^2}) \,dt \).
3. If \( \ddot{r}(t) = \begin{cases} \mathbf{i} - \mathbf{j} + 2\mathbf{k}, & \text{when } t = 1 \\ 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}, & \text{when } t = 2 \end{cases} \), show that \( \int_1^2 (\dddot{r} \cdot \frac{d\dot{r}}{dt}) \, dt = 10 \).

4. The acceleration of a particle at any time \( t \geq 0 \) is given by \( 12 \cos 2t \, \mathbf{i} - 8 \sin 2t \, \mathbf{j} + 16t \, \mathbf{k} \), the displacement and velocity are initially zero. Find the velocity and displacement at any time.

17.10 LINE INTEGRAL

Let \( \tilde{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k} \) defines a curve \( C \) joining points \( P_1 \) and \( P_2 \) where \( \tilde{r}(u) \) is the position vector of \( (x, y, z) \) and the value of \( u \) at \( P_1 \) and \( P_2 \) is \( u_1 \) and \( u_2 \), respectively.

Now if \( A(x, y, z) = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k} \) be vector function of defined position and continuous along \( C \), then the integral of the tangential component of \( A \) along \( C \) from \( P_1 \) to \( P_2 \) written as \( \int_{P_1}^{P_2} \tilde{A} \cdot d\tilde{r} \) is known as the line integral. Also in terms of Cartesian components, we have

\[
\int_{P_1}^{P_2} \tilde{A} \cdot d\tilde{r} = \int_{P_1}^{P_2} (A_1 \, dx + A_2 \, dy + A_3 \, dz)
\]

If \( \tilde{A} \) (vector function of the position) represents the force \( \tilde{F} \) on a particle moving along \( C \), then the line integral represents the work done by the force \( \tilde{F} \). If \( C \) is a simple closed curve, then the integral around \( C \) is generally written as

\[
\oint_{C} \tilde{A} \cdot d\tilde{r} = \oint_{C} A_1 \, dx + A_2 \, dy + A_3 \, dz
\]

In Fluid Mechanics and Aerodynamics, the above integral is called circulation of \( \tilde{A} \) about \( C \), where \( \tilde{A} \) represents the velocity of the fluid.

Let \( A = \text{grad} \, \phi \), then we have

\[
\oint_{C} \tilde{A} \cdot d\tilde{r} = \oint_{C} \left( \vec{\nabla} \phi \right) \cdot d\tilde{r} = \oint_{C} \left( \phi \frac{\partial \phi}{\partial x} \mathbf{i} + \phi \frac{\partial \phi}{\partial y} \mathbf{j} + \phi \frac{\partial \phi}{\partial z} \mathbf{k} \right) \cdot (dx \, \mathbf{i} + dy \, \mathbf{j} + dz \, \mathbf{k})
\]
\[= \int_\rho^Q \left( \frac{\partial \phi}{\partial x} \, dx + \frac{\partial \phi}{\partial y} \, dy + \frac{\partial \phi}{\partial z} \, dz \right)\]

\[= \int_\rho^Q d\phi = [\phi]^Q_\rho = \phi_Q - \phi_\rho \quad \text{... (1)}\]

We see that the integral \(\int_P^Q \vec{A}.d\vec{r}\) depends on the value of \(\phi\) at the end \(P\) and \(Q\) and not on the particular path. In case \(\phi\) is single valued and the integral is taken round a closed curve, the terminal points \(P\) and \(Q\) coincide and \(\phi_B = \phi_P\).

\[\text{[Because function } \phi \text{ is uniform]}\]

The integration along a closed curve is denoted by the sign of circle in the mid of the integral sign \(i.e\). for a uniform function, we have

\[\oint_C (\nabla \phi).d\vec{r} = 0 \quad \text{... (2)}\]

\textit{The converse of the above result is also true }i.e.\textit{ if there exists a vector }\vec{A}\textit{ and its integral round every closed curve in the region under consideration vanishes, then there exist a point function }\phi\textit{ such that }\vec{A} = \nabla \phi.\]

To prove this consider any closed curve \(ABCD\) such that the integral round it is zero, so integral along \(ABC\) must be equal to that along \(ADC\). Similarly, the integral along \(ABC\) must be equal to that along any curve joining \(A\) to \(C\), \(i.e.\) independent of the path from \(A\) to \(C\) with \(A\) be a fixed point and \(C\) a variable point. Then due to the fact that line integral is independent of the path chosen, the value of the line integral from \(A\) to \(C\) must be a scalar point function, say \(\phi\) \(i.e.\) \(\int_A^C \vec{A}.d\vec{r} = \phi\)

Now if \(d\phi\) is the increment in \(\phi\) due to a small displacement \(d\vec{r}\) of \(\vec{r}\), then we have \(d\phi = \vec{A}.d\vec{r}\)

But we already know that \(d\phi = \nabla \phi. d\vec{r}\), so \(\vec{A}.d\vec{r} = (\nabla \phi).d\vec{r}\),

\[\Rightarrow (\vec{A} - \nabla \phi).d\vec{r} = 0, \text{which is true for all } d\vec{r} \text{ and hence } \vec{A} = \nabla \phi.\]

The vector \(\vec{A}\) is called a \textbf{potential vector} (or gradient vector), and in cartesian component; the condition that \(\vec{A}.d\vec{r} = A_1 \, dx + A_2 \, dy + A_3 \, dz\) be a perfect differential can be thrown easily into the form

\[\frac{\partial A_2}{\partial z} - \frac{\partial A_3}{\partial y} = 0, \quad \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} = 0, \quad \frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x} = 0 \quad \text{... (3)}\]

\textbf{Circulation:} If \(\vec{A}\) represents the velocity of a fluid particle, then the line integral \(\int_C \vec{A}.d\vec{r}\) is called the circulation of \(\vec{A}\) along \(C\).

\textit{The vector point function }\vec{A}\textit{, is said to be irrotational in a region, if its circulation along every closed curve in the region is zero }i.e.\textit{ }\int_C \vec{A}.d\vec{r} = 0\]
Theorem: The necessary and sufficient condition for a vector point function $\mathbf{A}$ to be irrotational in a simply connected region is the curl $\mathbf{A} = 0$ at every point of the region.

Work: If $\mathbf{A}$ represents the force acting on a particle moving along an arc $PQ$ then the work done during the small displacement $d\mathbf{r}$ is equal to $\mathbf{A}.d\mathbf{r}$. Therefore, the total work done by $\mathbf{A}$ during the displacement from $P$ to $Q$ is given by the line integral $\int_P^Q \mathbf{A}.d\mathbf{r}$.

Example 48: Evaluate the line integral $\int [(x^2 + xy)dx + (x^2 + y^2)dy]$, where $C$ is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

Solution: Curve $C$ is square in the $xy$ plane where $z = 0$

$\mathbf{r} = xi + yj$ in $xy$ plane

$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$ \hspace{1cm} (1)

Now $\int_C \mathbf{F}.d\mathbf{r} = \int [(x^2 + xy)dx + (x^2 + y^2)dy]$

Path of the integration is shown in figure 7.12, it consists of lines $AB$, $BC$, $CD$ and $DA$. As curve $C$ is a square, then

On $AB$, $y = -1 \Rightarrow dy = 0$ and $x$ varies from $-1$ to $1$

On $BC$, $x = 1 \Rightarrow dx = 0$ and $y$ varies from $-1$ to $1$

On $CD$, $y = 1 \Rightarrow dy = 0$ and $x$ varies from $1$ to $-1$

On $DA$, $x = -1 \Rightarrow dx = 0$ and $y$ varies from $1$ to $-1$

$\int_C \mathbf{F}.d\mathbf{r} = \int_{AB}(x^2 - x)dx + \int_{BC}(1 + y^2)dy + \int_{CD}(x^2 + x)dx + \int_{DA}(1 + y^2)dy$

$= \int_{-1}^{1}(x^2 - x)dx + \int_{-1}^{1}(1 + y^2)dy + \int_{1}^{1}(x^2 + x)dx + \int_{-1}^{1}(1 + y^2)dy$

$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} + \int_{-1}^{1}(1 + y^2)dy + \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{1}^{1} - \int_{-1}^{1}(1 + y^2)dy$

$= \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2}\right) = 0$

Example 49: If $\mathbf{F} = 3xy\mathbf{i} - y^2\mathbf{j}$, evaluate $\int_C \mathbf{F}.d\mathbf{r}$ where $C$ is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution: Because the integration is performed in the $xy$-plane ($z = 0$), we take

$\mathbf{r} = xi + yj$ so that $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$

$\therefore \mathbf{F}.d\mathbf{r} = (3xy\mathbf{i} - y^2\mathbf{j}).(dx\mathbf{i} + dy\mathbf{j}) = 3xy\,dx - y^2\,dy$

On the curve $C$: $y = 2x^2$ from $(0,0)$ to $(1,2)$
\[ \vec{F} \cdot d\vec{r} = 3x(2x^2)dx - (2x^2)^24x \, dx = (6x^3 - 16x^5) \, dx \]

Also x varies from 0 to 1.

\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5) \, dx = \left[ \frac{6x^4}{4} - \frac{16x^6}{6} \right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6} \]

Note: If the curve is traversed in the opposite direction, that is from (1, 2) to (0, 0), the value of the integral would be \(\frac{7}{6}\).

Example 50: A vector field is given by \(\vec{F} = (\sin y)i + x(1 + \cos y)j\). Evaluate the line integral over the circular path given by \(x^2 + y^2 = a^2\), \(z = 0\). [PTU 2003]

**Solution:** The parametric equations of the circular path are \(x = a \cos t\), \(y = a \sin t\), \(z = 0\) where \(t\) varies from 0 to \(2\pi\). Since the particle moves in the xy-plane \((z = 0)\), we can take \(\vec{r} = i + yj\).

\[ \int_C \vec{F} \cdot d\vec{r} = \int_C [\sin y \, i + x(1 + \cos y)j] \cdot (dx \, i + dy \, j) \]

\[ = \int_C [\sin y \, dx + x(1 + \cos y)dy] \]

\[ = \int_C [(\sin y \, dx + x \cos y)dy + x \, dy] = \int_C d(x \sin y) + \int_C x \, dy \]

\[ = \int_0^{2\pi} a \cos t \sin(a \sin t) \, dt + a^2 \int_0^{2\pi} \cos^2 t \, dt \]

\[ = \frac{a^2}{2} \int_0^{2\pi} (1 + \cos 2t) \, dt = \frac{a^2}{2} \left[ t + \frac{\sin 2t}{2} \right]_0^{2\pi} = \frac{a^2}{2} (2\pi) = \pi a^2 \]

Example 51: Compute the line integral \(\int_C (y^2 \, dx - x^2 \, dy)\) about the triangle whose vertices are \((1, 0)\), \((0, 1)\) and \((-1, 0)\). [NIT Uttrakhand 2011]

**Solution:** Here the closed curve C is a triangle ABC.

On AB: Equation of line AB is \(y - 0 = \frac{1 - 0}{0 - 1} (x - 1) \Rightarrow y = 1 - x\)

\(\therefore \; dy = -dx\) and \(x\) varies from 1 to 0.

On BC: Equation of line BC is \(y - 1 = \frac{0 - 1}{-1 - 0} (x - 0) \Rightarrow y = 1 + x\)

\(\therefore \; dy = dx\) and \(x\) varies from 0 to -1.

On CA: \(y = 0\). Therefore, \(dy = 0\) and \(x\) varies from -1 to 1.

\[ \int_C (y^2 \, dx - x^2 \, dy) = \int_{AB} (y^2 \, dx - x^2 \, dy) + \int_{BC} (y^2 \, dx - x^2 \, dy) + \int_{CA} (y^2 \, dx - x^2 \, dy) \]

\[ = \int_1^0 [(1 - x)^2 \, dx - x^2 (-dx)] + \int_0^1 [(1 + x)^2 \, dx - x^2 \, dx] + \int_{-1}^0 \, dx \]
\[= \int_1^0 (2x^2 - 2x + 1)dx + \int_0^{x^{-1}} (2x + 1)dx + 0\]
\[= \left[\frac{2x^3}{3} - \frac{2x^2}{2} + x\right]_1^0 + \left[\frac{2x^2}{2} + x\right]_0^{-1} = \left(\frac{-2}{3} + 1 - 1\right) + (1 - 1) = -\frac{2}{3}\]

**Example 52:** If \(\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\), evaluate \(\int_C \vec{F}.d\vec{r}\) where

(i) \(C\) is the line joining the point \((0, 0, 0)\) to \((1, 1, 1)\)

(ii) \(C\) is given by \(x = t, y = t^2, z = t^3\) from the point \((0, 0, 0)\) to \((1, 1, 1)\).

**Solution:**
(i) Equation of line joining \((0,0,0)\) to \((1,1,1)\) is
\[
\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \quad \text{(say)}
\]
\[\therefore \text{Parametric equations of the line } C \text{ are } x = t, y = t, z = t; 0 \leq t \leq 1\]
\[\therefore \vec{r} = xt + y\hat{j} + zk = t\hat{i} + tf + t\hat{k}\]
\[
\Rightarrow \frac{d\vec{r}}{dt} = \hat{i} + \hat{j} + k
\]

Now \(\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\)
\[
\int_C \vec{F}.d\vec{r} = \int_C \left(\vec{F}.\frac{d\vec{r}}{dt}\right) dt
\]
\[
= \int_C [(3t^2 + 6t)\hat{i} - 14t^2\hat{j} + 20t^3\hat{k}].(\hat{i} + \hat{j} + \hat{k})dt
\]
\[
= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3) dt
\]
\[
= \left[\frac{3t^3}{3} + \frac{6t^2}{2} - \frac{14t^3}{3} + \frac{20t^4}{4}\right]_0^1 = (1 + 3 - 14 + 5) = \frac{13}{3}
\]

(ii) Here the curve \(C\) is given by \(x = t, y = t^2, z = t^3\) from the point \((0,0,0)\) to \((1,1,1)\)
\[\therefore \vec{r} = xt + y\hat{j} + zk = t\hat{i} + t^2\hat{j} + t^3\hat{k}\]
\[
\Rightarrow \frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}
\]

Now \(\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\)
\[
\int_C \vec{F}.d\vec{r} = \int_C \left(\vec{F}.\frac{d\vec{r}}{dt}\right) dt
\]
\[
= \int_C [9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k}].(\hat{i} + 2t\hat{j} + 3t^2\hat{k})dt
\]
\[
= \int_0^1 (9t^2 - 28t^6 + 60t^9) dt
\]
\[
= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1 = (3 - 4 + 6) = 5
\]

**Example 53:** Find the circulation of \(\vec{F}\) around the curve \(C\) where \(\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}\) and \(C\) is the circle \(x^2 + y^2 = 1, \ z = 0\).
Solution: Circulation of $\vec{F}$ along the curve $C$ is $\oint_C \vec{F} \cdot d\vec{r}$

Equation of circle is $x^2 + y^2 = 1, \ z = 0$

Its parametric equations are $x = \cos \theta, \ y = \sin \theta, \ z = 0$

Now $\vec{r} = xi + yj + zk = \cos \theta \hat{i} + \sin \theta \hat{j} + 0\hat{k}$ so that

$$d\vec{r} = (-\sin \theta \hat{i} + \cos \theta \hat{j} + 0\hat{k})d\theta$$

Also $\vec{F} = yi + zj + xk = \sin \theta \hat{i} + 0\hat{j} + \cos \theta \hat{k}$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \oint_C (\sin \theta \hat{i} + 0\hat{j} + \cos \theta \hat{k}) \cdot (-\sin \theta \hat{i} + \cos \theta \hat{j} + 0\hat{k})d\theta$$

(since along the circle, $\theta$ varies from 0 to $2\pi$)

$$= -\int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta = -\left[\frac{\theta - \sin 2\theta}{4}\right]_0^{2\pi}$$

$$= -\left[\left(\frac{2\pi}{2} - 0\right) - (0 - 0)\right] = -\pi$$

Example 54: Find the work done in moving a particle once round a circle $C$ in the $xy$ plane, if the circle has its centre at the origin and radius 2 units and the force field is given as

$$\vec{F} = (2x - y + 2z)i + (x + y - z)j + (3x - 2y - 5z)k.$$  

Solution: Equation of a circle having centre $(0,0)$ with radius 2 in $xy$ plane is $x^2 + y^2 = 4$. Parametric equations of this circle are $x = 2 \cos t, \ y = 2 \sin t, \ z = 0$.

Since integration is to be performed around a circle in $xy$ plane,

$\therefore \vec{r} = xi + yj = 2 \cos t \hat{i} + 2 \sin t \hat{j}$ \quad $\Rightarrow \frac{d\vec{r}}{dt} = -2 \sin t \hat{i} + 2 \cos t \hat{j}$

Work done, $\int_C \left(\vec{F} \cdot \frac{d\vec{r}}{dt}\right)dt$

$$= \int \{(2x - y + 2z)i + (x + y - z)j + (3x - 2y - 5z)k\} \cdot \{-2 \sin t \hat{i} + 2 \cos t \hat{j}\} \cdot dt$$

$$= \int \{(4 \cos t - 2 \sin t)i + (2 \cos t + 2 \sin t)j + (6 \cos t - 4 \sin t)k\} \cdot \{-2 \sin t \hat{i} + 2 \cos t \hat{j}\} \cdot dt$$

In moving round the circle, $t$ varies from 0 to $2\pi$

$\therefore \ \text{Work done} = \int_0^{2\pi} \{(4 \cos t - 2 \sin t)(-2 \sin t) + (2 \cos t + 2 \sin t)(2 \cos t)\} \cdot dt$

$$= \int_0^{2\pi} \{-8 \cos t \sin t + 4 \sin^2 t + 4 \cos^2 t + 4 \sin t \cos t\} \cdot dt$$

$$= \int_0^{2\pi} \{4 - 4 \sin^2 t \cos t\} \cdot dt = \left[4t - 4 \frac{\sin^2 t}{2}\right]_0^{2\pi}$$

$$= [(8\pi - 2 \sin^2 2\pi) - (0 - 0)] = 8\pi$$

ASSIGNMENT 6
1. Using the line integral, compute the work done by the force \( \vec{F} = (2y + 3)\hat{i} + (xz)\hat{j} + (yz - x)\hat{k} \), when it moves a particle from \((0,0,0)\) to \((2,1,1)\) along the curve \(x = 2t^2, y = t, z = t^3\).

2. Find the work done in moving a particle in the force field \( \vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k} \), along
   (a) the straight line from \((0,0,0)\) to \((2,1,1)\).
   (b) the curve defined by \(x^2 = 4y, 3x^3 = 8z\) from \(x = 0\) to \(x = 2\).

3. If \(C\) is a simple closed curve in the \(xy\) plane not enclosing the origin, show that \(\int_C \vec{F} \cdot d\vec{r} = 0\), where \(\vec{F} = \frac{y^2 - x^2}{x^2 + y^2}\).

4. If \(\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}\), evaluate \(\int_C \vec{F} \cdot d\vec{r}\) along the curve \(C\) in \(xy\)-plane, \(y = x^3\) from the point \((1,1)\) to \((2,8)\).

5. Evaluate \(\int_C (xy + z^2) \, ds\) where \(C\) is the arc of the helix \(x = \cos t, y = \sin t, z = t\) which joins the points \((1,0,0)\) and \((-1,0,\pi)\).

6. If \(\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}\), evaluate \(\int_C \vec{F} \times d\vec{r}\) along the curve \(x = \cos t, y = \sin t, z = 2\cos t\) from \(t = 0\) to \(t = \frac{\pi}{2}\).

17.11 SURFACE INTEGRALS AND FLUX

An integral which is to be evaluated over a surface is called a surface integral. Suppose \(S\) is a surface of finite area. Divide the area \(S\) into \(n\) sub-areas \(\delta S_1, \delta S_2, \ldots, \delta S_n\).

In each area \(\delta S_i\), choose an arbitrary point \(P_i(x_i, y_i, z_i)\). Let \(\varphi\) define a scalar point function over the area \(S\).

Now from the sum \(\sum_{i=1}^{n} \varphi(P_i) \delta S_i\), where \(\varphi(P_i) = \varphi(x_i, y_i, z_i)\)

Now let us take the limit of the sum as \(n \to \infty\), each sub-area \(\delta S_i\) reduces to a point and the limit if it exists is called the surface integral of \(\varphi\) over \(S\) and is denoted by \(\iint_S \varphi \, dS\).

**Note:** If \(S\) is piecewise smooth then the function \(\varphi(x, y, z)\) is continuous over \(S\) and then the limit exists and is independent of sub-divisions and choice of the point \(P_i\).

**Flux:** Suppose \(S\) is a piecewise smooth surface so that the vector function \(\vec{F}\) defined over \(S\) is continuous over \(S\). Let \(P\) be any point of the surface \(S\) and suppose \(\hat{n}\) is a unit vector at \(P\) in the direction of outward drawn normal to the surface \(S\) at \(P\). Then the component of \(\vec{F}\) along \(\hat{n}\) is \(\vec{F} \cdot \hat{n}\) and the integral of \(\vec{F} \cdot \hat{n}\) over \(S\) is called the surface integral of \(\vec{F}\) over \(S\) and is denoted by \(\iint_S \vec{F} \cdot \hat{n} \, dS\). It is also called flux of \(\vec{F}\) over \(S\).

**Different Forms of Surface Integral**

(i) Flux of \(\vec{F}\) over \(S\) = \(\iint_S \vec{F} \cdot \hat{n} \, dS\) \hspace{1cm} \ldots \hspace{1cm} (1)
Now let \( \overrightarrow{dS} \) denote a vector (called vector area) whose magnitude is that of differential of surface area i.e., \( dS \) and whose direction is that of \( \hat{n} \). Then clearly
\[
\overrightarrow{dS} = \hat{n} dS
\]
From (1), flux of \( \vec{F} \) over \( S \) is
\[
\iint_S \vec{F} \cdot d\overrightarrow{S} \quad \hdots (2)
\]
(ii) Suppose outward drawn normal to the surface \( S \) at \( P \) makes angles \( \alpha, \beta, \gamma \) with the positive direction of axes and if \( l, m, n \) denote the direction cosines of this outward drawn normal, then
\[
 l = \cos \alpha, \quad m = \cos \beta, \quad n = \cos \gamma
\]
Therefore, \( \hat{n} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k} \)
If \( \vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \) then \( \vec{F} \cdot \hat{n} = F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma \)
\( \therefore \) From (1), flux of \( \vec{F} \) over \( S \) is
\[
\iint_S (F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma) \; dS \quad \hdots (3)
\]
Now \( dS \cos \alpha \) is the projection of area \( dS \) on the \( yz \) plane, therefore \( dS \cos \alpha = dydz \).
Similarly \( dS \cos \beta \) and \( dS \cos \gamma \) are the projections of the area \( dS \) on the \( zx \) and \( xy \) plane respectively and therefore \( dS \cos \beta = dzdx \), \( dS \cos \gamma = dx dy \).
\( \therefore \) From (3), \( \iint_S \vec{F} \cdot \hat{n} \; dS = \iint_{R_1} \vec{F}_1 dydz + \iint_{R_2} \vec{F}_2 dzdx + \iint_{R_3} \vec{F}_3 dx dy \) \( \hdots (4) \)

**Note:** In order to evaluate surface integral it is convenient to express them as double integrals by taking the projection of surface \( S \) on one of the coordinate planes. This will happen only if any line perpendicular to co-ordinate plane chosen meets the surface in one point and not more than one point. Surface \( S \) is divided into sub surfaces, if above requirement is not met, so that sub surfaces may satisfy the above requirement.

(iii) Suppose surface \( S \) is such that any line perpendicular to \( xy \) plane does not meet \( S \) in more than one point. Let the equation of surface \( S \) be \( Z = h(x, y) \).
Let \( R_1 \) denotes the orthogonal projection of \( S \) on the \( xy \) plane. Then projection of \( dS \) on the \( xy \) plane is \( dS \cos \gamma \), where \( \gamma \) is the acute angle which the normal to the surface \( S \) makes with positive direction of \( Z \)-axis.
\( \therefore \) \( dS \cos \gamma = dx dy \) \( \hdots (5) \)
But \( \cos \gamma = \frac{[\hat{n}, \hat{k}]}{[\hat{n}]} = \frac{|\hat{n}, \hat{k}|}{|\hat{n}|} \)
Therefore from (5), \( dS = \frac{dx \; dy}{|\hat{n}, \hat{k}|} \) \( \hdots (6) \)
Thus \( \iint_S \vec{F} \cdot \hat{n} \; dS = \iint_{R_1} \vec{F}_1 \cdot \hat{n} \frac{dy \; dz}{|\hat{n}, \hat{k}|} \) \( \hdots (7) \)
Similarly we have, \( \iint_S \vec{F} \cdot \hat{n} \; dS = \iint_{R_2} \vec{F}_2 \cdot \hat{n} \frac{dz \; dx}{|\hat{n}, \hat{j}|} \) \( \hdots (8) \)
\( \iint_S \vec{F} \cdot \hat{n} \; dS = \iint_{R_3} \vec{F}_3 \cdot \hat{n} \frac{dx \; dy}{|\hat{n}, \hat{i}|} \) \( \hdots (9) \)
where \( R_2, R_3 \) are the projections of \( S \) on \( zx \) and \( xy \) planes, respectively.
Example 55: Evaluate \( \iiint_S \vec{F} \cdot \hat{n} \, dS \) where \( \vec{F} = z \hat{i} + x \hat{j} + 3yz \hat{k} \) and \( S \) is the surface of the cylinder \( x^2 + y^2 = 16 \) included in the first octant between \( z = 0 \) and \( z = 5 \).

Solution: A vector normal to the surface \( S \) is given by
\[
\hat{n} = \nabla (x^2 + y^2) = 2x \hat{i} + 2y \hat{j}
\]
\[
\therefore \quad \hat{n} = \text{unit vector normal to surface at any point } (x, y, z) = \frac{2x \hat{i} + 2y \hat{j}}{\sqrt{4x^2 + 4y^2}} \quad \cdots (1)
\]
\[
\therefore \quad x^2 + y^2 = 16, \quad \text{therefore } \hat{n} = \frac{2x \hat{i} + 2y \hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x^2 + 2y^2}{8} = \frac{x}{4} \hat{i} + \frac{y}{4} \hat{j} \quad \cdots (3)
\]

Now \( \iiint_S \vec{F} \cdot \hat{n} \, dS = \iint_R \vec{F} \cdot \hat{n} \frac{dx \, dz}{|n|} \)

(projection on \( xy \) plane can’t be taken as the surface \( S \) is perpendicular to \( xy \) plane)
\[
= \iint_R \left( \frac{xz}{4} + \frac{xy}{4} \right) \frac{dx \, dz}{\frac{\sqrt{16 - x^2}}{2}} = \iint_R \left( \frac{xz}{4} \right) + x \right) \, dx \, dz, \quad (\text{since from (3), } \hat{n} \cdot \hat{j} = \frac{y}{4})
\]
\[
= \int_0^5 \int_{x=0}^4 \left( \frac{xz - 2x}{\sqrt{16 - x^2}} + x \right) \, dx \, dz
\]
\[
= \int_0^5 \left[ -\frac{z}{2} \cdot \frac{16 - x^2}{x^2} + \frac{x^2}{2} \right]_{x=0}^4 \, dz = \int_0^5 (4z + 8) \, dz = \left[ \frac{4z^2}{2} + 8z \right]_{z=0}^5 = 90
\]

Example 56: Evaluate \( \iiint_S \vec{F} \cdot \hat{n} \, dS \) where \( \vec{F} = 6z \hat{i} - 4 \hat{j} + y \hat{k} \) and \( S \) is the portion of the plane \( 2x + 3y + 6z = 12 \) in the first octant.

Solution: Vector normal to surface \( S \) is given by
\[
\nabla (2x + 3y + 6z) = 2 \hat{i} + 3 \hat{j} + 6 \hat{k}
\]
\[
\therefore \quad \hat{n} = \text{unit vector normal to surface at any point } (x, y, z) = \frac{2 \hat{i} + 3 \hat{j} + 6 \hat{k}}{\sqrt{4 + 9 + 36}} = \frac{2}{\sqrt{49}} \hat{i} + \frac{3}{\sqrt{49}} \hat{j} + \frac{6}{\sqrt{49}} \hat{k}
\]

Now \( \vec{F} \cdot \hat{n} = \left( 6z \hat{i} - 4 \hat{j} + y \hat{k} \right) \cdot \left( \frac{2}{7} \hat{i} + \frac{3}{7} \hat{j} + \frac{6}{7} \hat{k} \right) = \frac{12}{7} z - \frac{12}{7} + \frac{6}{7} y
\]

Taking projection on \( xy \) plane
\[
\iiint_S \vec{F} \cdot \hat{n} \, dS = \iint_R \vec{F} \cdot \hat{n} \frac{dy \, dx}{|n|} = \iint_R \vec{F} \cdot \hat{n} \frac{dy \, dx}{6/7} \quad \cdots (1)
\]

where \( R \) is the region of projection of \( S \) on \( xy \) plane. \( R \) is bounded by \( x \)-axis, \( y \)-axis and the line \( 2x + 3y = 12, z = 0 \). In order to evaluate double integral in (1), \( y \) varies from 0 to 4 and \( x \) varies from 0 to \( \frac{12 - 3y}{2} \). Therefore from (1)
\[
\iiint_S \vec{F} \cdot \hat{n} \, dS = \int_{y=0}^4 \int_{x=0}^{\frac{12 - 3y}{2}} [2z - 2 + y] \, dx \, dy,
\]
\[
= \int_{y=0}^4 \int_{x=0}^{\frac{12 - 3y}{2}} \left[ 2 \left( 2 - \frac{x}{3} - \frac{y}{2} \right) - 2 + y \right] \, dx \, dy
\]
\[ = \int_{y=0}^{4} \int_{x=0}^{12-3y} \left[ 2 - \frac{2x}{3} \right] dx \, dy = \int_{0}^{4} \left[ 2x - \frac{x^2}{3} \right]_{x=0}^{12-3y} dy \]

\[ = \int_{0}^{4} \left[ 2 \times \frac{12-3y}{2} - \frac{1}{3} \left( \frac{12-3y}{2} \right)^2 \right] dy \]

\[ = \left[ \frac{(12-3y)^2}{-6} + \frac{(12-3y)^3}{108} \right]_{y=0}^{4} = \frac{144}{6} - \frac{1728}{108} = 24 - 16 = 8 \]

**Example 57:** \( \int_S \varphi \, \mathbf{n} \cdot dS \) where \( \varphi = \frac{3}{8} xyz \) and \( S \) is the surface of cylinder \( x^2 + y^2 = 16 \) included in the first octant between \( z = 0 \) to \( z = 5 \).

**Solution:** A vector normal to the surface \( S \) is given by

\[ \mathbf{n} = \nabla (x^2 + y^2) = 2x \mathbf{i} + 2y \mathbf{j} \]

\[ \therefore \mathbf{n} = \text{unit vector normal to surface } S \text{ at any point } (x, y, z) = \frac{2x \mathbf{i} + 2y \mathbf{j}}{\sqrt{4x^2 + 4y^2}} \]

\[ \mathbf{n} = \frac{2x \mathbf{i} + 2y \mathbf{j}}{2} = \frac{x}{4} \mathbf{i} + \frac{y}{4} \mathbf{j} \quad \text{[} \because \; x^2 + y^2 = 16 \]}

Now \( \int_S \varphi \, \mathbf{n} \cdot dS = \int_R \frac{3}{8} xyz \left( \frac{x}{4} \mathbf{i} + \frac{y}{4} \mathbf{j} \right) \frac{dx \, dy}{|\mathbf{n}|} \)

where \( R \) is the region of projection of \( S \) on \( zx \) plane. Therefore, from (1)

\[ \int_S \varphi \, \mathbf{n} \cdot dS \]

\[ = \int_{x=0}^{5} \int_{y=0}^{4} \left( \frac{3}{32} x^2 yz \mathbf{i} + \frac{3}{32} xy^2 z \mathbf{j} \right) \frac{dx \, dy}{y/4} , \quad \text{where} \; y^2 = 16 - x^2 \]

\[ = \int_{x=0}^{5} \int_{y=0}^{4} \left( \frac{3}{8} x^2 z \mathbf{i} + \frac{3}{8} xz \sqrt{(16 - x^2)} \mathbf{j} \right) dx \, dz \]

\[ = \frac{3}{8} \int_{z=0}^{5} \left( \frac{x^2 z}{3} \mathbf{i} - \frac{x}{2} \left( \frac{16-x^2}{3} \right)^{3/2} \mathbf{j} \right)_{x=0}^{4} \, dz \]

\[ = \frac{3}{8} \int_{z=0}^{5} \left( \frac{64}{3} z \mathbf{i} + \frac{64}{3} z \mathbf{j} \right) dz = 8 \left[ \frac{z^2}{2} \mathbf{i} + \frac{z^2}{2} \mathbf{j} \right]_{z=0}^{5} = 100 \mathbf{i} + 100 \mathbf{j} \]

**Example 58:** Evaluate \( \int_S \mathbf{F} \cdot dS \) where \( \mathbf{F} = x \mathbf{i} - (z^2 - xz) \mathbf{j} - xy \mathbf{k} \) and \( S \) is the triangular surface with vertices \( (2, 0, 0) \), \( (0, 2, 0) \) and \( (0, 0, 4) \).

**Solution:** The triangular surface \( S \) with vertices \( (2,0,0) \), \( (0,2,0) \), and \( (0,0,4) \) is given by the equation

\[ \frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1 \quad \Rightarrow \quad 2x + 2y + z = 4 \quad \text{… (1)} \]

Vector normal to surface \( S \) is given by \( \nabla (2x + 2y + z) = 2 \mathbf{i} + 2 \mathbf{j} + \mathbf{k} \)

\[ \therefore \mathbf{n} = \text{unit vector normal to surface } S \text{ at any point } (x, y, z) = \frac{2 \mathbf{i} + 2 \mathbf{j} + \mathbf{k}}{\sqrt{4 + 4 + 1}} = \frac{2}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} + \frac{1}{3} \mathbf{k} \]

Now \( \mathbf{F}, \mathbf{n} = (x \mathbf{i} - (z^2 - xz) \mathbf{j} - xy \mathbf{k}). \left( \frac{2}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} + \frac{1}{3} \mathbf{k} \right) = \frac{2}{3} x - \frac{2}{3} (z^2 - xz) - \frac{1}{3} xy \)

Taking projection on \( xy \) plane
\[ \iint_S \vec{F} \cdot \hat{n} \, dS = \int_R \vec{F} \cdot \hat{n} \frac{dx \, dy}{\| \hat{n} \|^{1/3}} \]  

where \( R \) is the region of projection of \( S \) on \( xy \) plane. \( R \) is bounded by \( x \)-axis, \( y \)-axis and the line \( 2x + 2y = 4 \) i.e., \( x + y = 2, \) \( z = 0. \) In order to integrate double integral in (2), \( y \) varies from 0 to 2 and \( x \) varies from 0 to \( 2 - y. \) Therefore from (2)

\[
\begin{align*}
\iint_S \vec{F} \cdot \hat{n} \, dS &= \int_{y=0}^{2} \int_{x=0}^{2-y} \left[ \frac{2}{3} x - \frac{2}{3} (z^2 - zx) - \frac{1}{3} xy \right] \, dx \, dy \\
&= \frac{1}{3} \int_{y=0}^{2} \int_{x=0}^{2-y} \left[ 2x - 2 \{16 + 4x^2 + 4y^2 - 16x + 8xy - 16y - 4x + 2x^2 + 2xy\} - xy \right] \, dx \, dy \\
&= \frac{1}{3} \int_{y=0}^{2} \int_{x=0}^{2-y} (-12x^2 - 8y^2 + 42x + 32y - 21xy - 32x) \, dx \, dy \\
&= \frac{1}{3} \int_{y=0}^{2} \left[ -4x^3 - 8xy^2 + 21x^2 + 32xy - 21 \frac{x^2y}{2} - 32x \right]_{x=0}^{2-y} \, dy \\
&= \frac{1}{3} \int_{y=0}^{2} \left[ -4(2 - y)^3 - 8(2 - y)y^2 + 21(2 - y)^2 + 32(2 - y)y - 21 \frac{(2-y)^2y}{2} - 32(2 - y) \right] \, dy \\
&= \frac{1}{3} \int_{y=0}^{2} \left( \frac{3}{2} y^3 + 9y^2 + 18y - 12 \right) \, dy = 38
\end{align*}
\]

Example 59: Evaluate \( \int_S \vec{F} \cdot \hat{n} \, ds \) where \( \vec{F} = 2x^2 \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k} \) and \( S \) is closed surface of the region in the first octant bounded by the cylinder \( y^2 + z^2 = 9 \) and the planes \( x = 0, \) \( y = 0 \) and \( z = 0. \)

**Solution:** The given closed surface \( S \) is piecewise smooth and is comprised of \( S_1 \) – the rectangular face OAEB in \( xy \)-plane; \( S_2 \) – the rectangular face OADC in \( xz \)-plane; \( S_3 \) – the circular quadrant ABC in \( yz \)-plane; \( S_4 \) – the circular quadrant AED and \( S_5 \) – the curved surface BCDE of the cylinder in the first octant (see Fig. 17.16).

\[
\therefore \quad \int_S \vec{F} \cdot \hat{n} \, ds = \int_{S_1} \vec{F} \cdot \hat{n} \, ds + \int_{S_2} \vec{F} \cdot \hat{n} \, ds + \int_{S_3} \vec{F} \cdot \hat{n} \, ds \\
+ \int_{S_4} \vec{F} \cdot \hat{n} \, ds + \int_{S_5} \vec{F} \cdot \hat{n} \, ds \quad \ldots (1)
\]

Now \( \int_{S_1} \vec{F} \cdot \hat{n} \, ds = \int_{S_1} (2x^2 y \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k}) \cdot \hat{k} \, ds = -4 \int_{S_1} xz^2 \, ds = 0 \) (as \( z = 0 \) in \( xy \)-plane)

Similarly, \( \int_{S_2} \vec{F} \cdot \hat{n} \, ds = 0 \) and \( \int_{S_3} \vec{F} \cdot \hat{n} \, ds \)

\[
\int_{S_4} \vec{F} \cdot \hat{n} \, ds = \int_{S_4} (2x^2 y \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k}) \cdot \hat{i} \, ds = \int_{S_4} 2x^2 y \, ds \\
= \int_{0}^{3} \int_{0}^{\sqrt{9 - x^2}} 8y \, dy \, dz = 4 \int_{0}^{3} (9 - z^2) \, dz = 72
\]

To find \( \hat{n} \) in \( S_5 \), we note that \( \nabla (y^2 + z^2 - 9) = 2y \hat{j} + 2z \hat{k}, \)

Implying \( \hat{n} = \frac{2y \hat{j} + 2z \hat{k}}{\sqrt{4(y^2 + z^2)}} = \frac{y \hat{j} + z \hat{k}}{\sqrt{3}} \) and \( \| \hat{n} \| \hat{k} = \frac{z}{\sqrt{3}} \) so that \( ds = dx \, dy / (z/3) \) (as \( y^2 + z^2 = 9 \))

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\[
\int_S \vec{F} \cdot \hat{n} \, ds = \int_0^2 \int_0^3 \left( -\frac{y^3 + 4xz^3}{3} \right) \, dx \, dy / (z/3) = \int_0^2 \int_0^3 \left( \frac{y^3}{z} + 4xz^2 \right) \, dx \, dy
\]

Now putting \( y = 3 \sin \theta \), \( z = 3 \cos \theta \) \( \therefore dy = 3 \cos \theta \, d\theta \)

\[
\int_0^2 \int_0^3 \left[ -\frac{27 \sin^3 \theta}{3 \cos \theta} + 4x (9 \cos^2 \theta) \right] 3 \cos \theta \, d\theta \, dx
\]

**ASSIGNMENT 7**

1. If velocity vector is \( \vec{F} = y \hat{i} + 2 \hat{j} + xz \hat{k} \) m/sec., show that the flux of water through the parabolic cylinder \( y = x^2 \), \( 0 \leq x \leq 3 \), \( 0 \leq z \leq 2 \) is \( 69 \) m\(^3\)/sec.

2. Evaluate \( \iint_S \vec{F} \cdot \hat{n} \, dS \) where \( \vec{F} = (x + y^2) \hat{i} - 2x \hat{j} + 2yz \hat{k} \) and S is the surface of the plane \( 2x + y + 2z = 6 \) in the first octant.

3. If \( \vec{F} = 4xz \hat{i} - y^2 \hat{j} + yz \hat{k} \); evaluate \( \iint_S \vec{F} \cdot \hat{n} \, dS \), where S is the surface of the cube bounded by \( x = 0 \), \( x = 1 \), \( y = 0 \), \( y = 1 \), \( z = 0 \), \( z = 1 \).

4. If \( \vec{F} = 2y \hat{i} - 3 \hat{j} + x^2 \hat{k} \) and S is the surface of the parabolic cylinder \( y^2 = 8x \) in the first octant bounded by the planes \( y = 4 \) and \( z = 6 \), show that \( \iint_S \vec{F} \cdot \hat{n} \, ds = 132 \).

5. Evaluate \( \int_S \vec{F} \cdot \hat{n} \, ds \) where \( \vec{F} = 6z \hat{i} - 4 \hat{j} + y \hat{k} \) and S is the portion of the plane \( 2x + 3y + 6z = 12 \) in the first octant.

**17.12 VOLUME INTEGRALS**

Suppose \( V \) is the volume bounded by a surface \( S \). Divide the volume \( V \) into sub-volumes \( \delta V_1, \delta V_2, \ldots, \delta V_n \). In each \( \delta V_i \), choose an arbitrary point \( P_i \) whose coordinates are \( (x_i, y_i, z_i) \). Let \( \varphi \) be a single valued function defined over \( V \). Form the sum \( \sum \varphi (P_i) \, \delta V_i \), where \( \varphi (P_i) = \varphi (x_i, y_i, z_i) \).

Now let us take the limit of the sum as \( n \to \infty \), then the limit, if exists, is called the volume integral of \( \varphi \) over \( V \) and is denoted as \( \iiint_V \varphi \, dV \).

Likewise if \( \vec{F} \) is a vector point function defined in the given region of volume \( V \) then vector volume integral of \( \vec{F} \) over \( V \) is \( \iiint_V \vec{F} \, dV \).

**Note:** Above volume integral becomes \( \iiint_V \varphi \, dx \, dy \, dz \) if we subdivide the volume \( V \) into small cuboids by drawing lines parallel to three co-ordinate axes because in that case \( dV = dx \, dy \, dz \).

**Example 60:** If \( \vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k} \), evaluate \( \iiint_V \nabla \times \vec{F} \, dV \) where \( V \) is the region bounded by the co-ordinate planes and the plane \( 2x + 2y + z = 4 \).

**Solution:** Consider
\[ \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = 0 \mathbf{i} + j - 2y \mathbf{k} \]

Region bounded by \(2x + 2y + z = 4\) and coordinate planes such that

\[
\begin{align*}
2x & \leq 4, \quad 2x + 2y \leq 4, \quad 2x + 2y + z \leq 4 \\
i.e. \quad x & \leq 2, \quad y \leq 2 - x, \quad z \leq 4 - 2x - 2y
\end{align*}
\]

\[
\therefore \iiiint_V \nabla \times \mathbf{F} \, dV = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (j - 2y \mathbf{k}) \, dz \, dy \, dx
\]
\[
= \int_{x=0}^{2} \int_{y=0}^{2-x} \left( z \mathbf{j} - 2yz \mathbf{k} \right)_{z=0}^{z=4-2x-2y} \, dy \, dx
\]
\[
= \int_{x=0}^{2} \left[ (4y - 2xy - y^2) \mathbf{j} - \left( 4y^2 - 2xy^2 - \frac{4}{3} y^3 \right) \mathbf{k} \right]_{y=0}^{y=2-x} \, dx
\]
\[
= \int_{x=0}^{2} \left[ (4x - 2x^2 + \frac{4}{3} x^3) \mathbf{j} - \left( -\frac{2}{3} x^3 + 4x^2 - 8x + \frac{16}{3} \right) \mathbf{k} \right] \, dx
\]
\[
= \left[ 4x - 2x^2 + \frac{4}{3} x^3 \right]_{x=0}^{x=2} \mathbf{j} - \left[ -\frac{4}{6} x^4 + \frac{4}{3} x^3 - 4x^2 + \frac{16}{3} x \right]_{x=0}^{x=2} \mathbf{k}
\]
\[
= \frac{8}{3} \mathbf{j} - \frac{8}{3} \mathbf{k} = \frac{8}{3} (\mathbf{j} - \mathbf{k})
\]

**ASSIGNMENT 8**

1. Evaluate \(\iiint_V \varphi \, dV\) where \(\varphi = 45x^2y\) and \(V\) is the region bounded by the planes \(4x + 2y + z = 8, \ x = 0, \ y = 0, \ z = 0\).

2. If \(\mathbf{F} = 2xz \mathbf{i} - x \mathbf{j} + y^2 \mathbf{k}\); evaluate \(\iiint_V \mathbf{F} \, dV\) where \(V\) is the region bounded by the planes \(x = 0, \ y = 0, \ x = 2, \ y = 6, \ z = x^2, \ z = 4\).

**17.13 STOKES’ THEOREM (Relation between Line and Surface Integral)**

**Statement:** Let \(S\) be a piecewise smooth open surface bounded by a piecewise smooth simple curve \(C\). If \(\mathbf{F}(x,y,z)\) be a continuous vector function which has continuous first partial derivative in a region of space which contains \(S\), then

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot \hat{n} \, dS,
\]

where \(\hat{n}\) is the unit normal vector at any point of \(S\) and \(C\) is traversed in positive direction.

Direction of \(C\) is positive if an observer walking on the boundary of \(S\) in this direction with its head pointing in the direction of outward normal \(\hat{n}\) to \(S\) has the surface on the left.

We may put the statement of Stoke’s theorem in words as under:
The line integral of the tangential component of a vector \( \vec{f} \) taken around a simple closed curve \( C \) is equal to the surface integral of normal component of curl of \( \vec{f} \) taken over \( S \) having \( C \) as its boundary.

**Stoke’s Theorem in Cartesian Form:**

a) **Cartesian Form of Stoke’s Theorem in Plane (or Green’s Theorem in Plane)**

Choose system of coordinate axes such that the plane of the surface is in \( xy \) plane and normal to the surface \( S \) lies along the \( z \)-axis. Normal vector is constant in this case.

Suppose \( \vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \)

\[
\oint_C \vec{f} \cdot d\vec{r} = \iint_S \text{curl} \vec{f} \cdot \hat{n} \, dS
\]

But tangent at any point lies in the \( xy \) plane, so \( \frac{dz}{ds} = 0 \)

\[
\oint_C \vec{f} \cdot d\vec{r} = \int_C \left( f_1 \frac{dx}{ds} + f_2 \frac{dy}{ds} \right) ds
\]

Now

\[
\iint_S \text{curl} \vec{f} \cdot \hat{n} \, dS = \iint_S \text{curl} \vec{f} \cdot \hat{k} \, dS
\]

(Here normal is along \( Z \)-axis)

Using (1) and (2), Stoke’s theorem is

\[
\oint_C (f_1 \, dx + f_2 \, dy) = \iint_S \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dxdy
\]

b) **Cartesian Form of Stoke’s Theorem in Space**

Suppose \( \vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \) and \( \hat{n} \) is an outward drawn normal unit vector of \( S \) making angles \( \alpha, \beta, \gamma \) with positive direction of axes.

\[
\hat{n} = \cos \alpha \, \hat{i} + \cos \beta \, \hat{j} + \cos \gamma \, \hat{k}
\]

Now,

\[
\nabla \times \vec{f} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
f_1 & f_2 & f_3
\end{vmatrix}
\]

i.e.

\[
\text{curl} \vec{f} = \left[ \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i} + \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \hat{j} + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \hat{k} \right]
\]

\[
\text{curl} \vec{f} \cdot \hat{n} = \left( \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial z} \right) \cos \alpha + \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \cos \beta + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \cos \gamma
\]

\[
\oint_C \vec{f} \cdot d\vec{r} = (f_1 \, dx + f_2 \, dy + f_3 \, dz)
\]

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or \[ \vec{f} \cdot d\vec{r} = (f_1 \, dx + f_2 \, dy + f_3 \, dz) \]

Then Stoke’s theorem is

\[ \oint_C (f_1 \, dx + f_2 \, dy + f_3 \, dz) = \int_S \left[ \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \cos \alpha + \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \cos \beta + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \cos \gamma \right] dS \]

Example 61: Verify Stoke’s theorem for \( \vec{f} = (2x - y)i - yz^2j - y^2zk \) when \( S \) is the upper half of the surface of the sphere \( x^2 + y^2 + z^2 = 1 \) and \( C \) is its boundary.

Solution: The boundary \( C \) of the upper half of the sphere \( S \) is circle in the \( xy \)-plane. Therefore, parametric equations of \( C \) are \( x = \cos t, \ y = \sin t, \ z = 0 \) when \( 0 \leq t \leq 2\pi \)

Now, \( \oint_C \vec{f} \cdot d\vec{r} = \oint_C (f_1 \, dx + f_2 \, dy + f_3 \, dz) \)

\[ \begin{align*}
&= \oint_C (2x - y)dx - yz^2dy - y^2zd\zeta = \oint_C (2 \cos t - \sin t)(-\sin t) \, dt \\
&[\because \ x = \cos t, \ y = \sin t \ \text{and other terms of integrand become zero as} \ z = 0]
\end{align*} \]

\[ \begin{align*}
&= \int_0^{2\pi} (2 \cos t)(-\sin t) + \sin^2 t \, dt = \left[ 2 \cos^2 \frac{\pi}{2} \right]_0^{2\pi} + \int_0^{2\pi} \sin^2 t \, dt \\
&= (1 - 1) + 4 \int_0^{\pi/2} \sin^2 t \, dt \quad \text{(Property of definite integral)} \\
&= 0 + 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi \\
&\quad \ldots (1)
\end{align*} \]

Now, \( \text{curl} \, \vec{f} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x - y & -yz^2 & -y^2z \\
\end{vmatrix} = (-2yz + 2yz)i + (0 - 0)j + (0 + 1)k = \hat{k} \)

Now, \( \iint_S \text{curl} \, \vec{f} \cdot \hat{n} \, dS = \iint_R \hat{k} \cdot \hat{n} \, dS \)

\[ = \iint_R \hat{k} \cdot \hat{n} \, \frac{dx \, dy}{|\hat{n} \cdot \hat{k}|} \quad \text{[where} \ R \ \text{is the projection of} \ S \ \text{on} \ xy \ - \text{plane]} \]

\[ = \iint_R \, dx \, dy \]

Now projection of \( S \) on \( xy \) plane is circle \( x^2 + y^2 = 1 \).

\[ \begin{align*}
&= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \, dy = 4 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} dx \, dy \quad \text{[By definite integral]} \\
&= 4 \int_{x=0}^{1} \sqrt{1-x^2} \, dx = 4 \left[ \frac{x \sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_{x=0}^{1} \\
&= 4 \left[ \frac{1}{2} \sin^{-1} 1 \right] = 4 \times \frac{\pi}{4} = \pi \\
&\quad \ldots (2)
\end{align*} \]

From (1) and (2), Stoke’s theorem is verified.

Example 62: Verify Stoke’s theorem for the function \( \vec{f} = (x^2 + y^2) \, \hat{i} - 2xy \, \hat{j} \) taken round the rectangle bounded by \( x = \pm a, \ y = 0, \ y = b \). \quad [\text{KUK 2006}]
Solution: Given \( \vec{F} = (x^2 + y^2) \hat{i} - 2xy \hat{j} \)

Therefore, \( \vec{F} \cdot d\vec{r} = [(x^2 + y^2) \hat{i} - 2xy \hat{j}] \cdot [dx \hat{i} + dy \hat{j}] = (x^2 + y^2) \, dx - 2xy \, dy \)

\[
\begin{align*}
\int_{DA} & (x^2 + y^2) \, dx - 2xy \, dy = \int_{DA} -2(-a)y \, dy = \int_{y=-a}^{y=a} 2ay \, dy = [ay^2]_{-a}^{a} = -ab^2 \\
\int_{AB} & (x^2 + y^2) \, dx - 2xy \, dy = \int_{AB} x^2 \, dx = \int_{-a}^{a} x^2 \, dx = \frac{a^3}{3} \\
\int_{BC} & (x^2 + y^2) \, dx - 2xy \, dy = \int_{BC} (-2ay) \, dy = \int_{y=-b}^{y=b} (-2ay) \, dy = -ab^2 \\
\int_{CD} & (x^2 + y^2) \, dx - 2xy \, dy = \int_{CD} (x^2 + b^2) \, dx = \int_{x=a}^{x=-a} (x^2 + b^2) \, dx \\
& = \left[ \frac{x^3}{3} + b^2x \right]_{-a}^{a} = -2 \left( \frac{a^3}{3} + b^2a \right)
\end{align*}
\]

Substituting these values in (1), we get

\[
\Phi_C \vec{F} \cdot d\vec{r} = -ab^2 + \frac{2}{3} a^3 - ab^2 - 2 \left( \frac{a^3}{3} + b^2a \right) = -4ab^2
\]

Now, \( \text{curl} \, \vec{F} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\hat{i} & \hat{j} & \hat{k} \\
x^2 + y^2 & -2xy & 0
\end{vmatrix} = 0 \hat{i} + 0 \hat{j} + (2y - 2y) \hat{k} = -4y \hat{k} \)

Since Surface lies in xy-plane, therefore \( \hat{n} = \hat{k} \).
\[ \mathbf{\nabla} \times \mathbf{f} \cdot \mathbf{n} \, dS = \int_S (-4y \mathbf{\hat{k}}) \cdot \mathbf{\hat{k}} \, dS = \int_{y=a}^{b} \int_{x=-a}^{a} -4y \, dy \, dx = -4ab^2 \quad \ldots (3) \]

Hence from (2) and (3), theorem is verified.

Example 63: Evaluate by Stoke’s theorem \( \oint_C (yz \, dx + xz \, dy + xydz) \), where \( C \) is the curve \( x^2 + y^2 = 1, \ z = y^2 \).

**Solution:** \( \oint_C (yz \, dx + xz \, dy + xydz) = \oint_C (yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) \)

\[ = \oint_C \mathbf{\tilde{f}} \cdot d\mathbf{\hat{r}}, \quad \text{where } \mathbf{\tilde{f}} = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} \]

Now, \( \mathbf{\nabla} \times \mathbf{\tilde{f}} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \\ \end{vmatrix} = (x-x)i + (y-y)j + (z-z)k = 0 \)

\[ \therefore \oint_C \mathbf{\tilde{f}} \cdot d\mathbf{\hat{r}} = \oint_S \mathbf{\nabla} \times \mathbf{\tilde{f}} \cdot \mathbf{n} \, dS = 0 \quad \left[ \because \mathbf{\nabla} \times \mathbf{\tilde{f}} = 0 \right] \]

Example 64: Evaluate \( \oint_C \mathbf{\tilde{f}} \cdot d\mathbf{\hat{r}} \) by Stoke’s theorem, where \( \mathbf{\tilde{f}} = y^2 \mathbf{i} + x^2 \mathbf{j} - (x+z) \mathbf{k} \) and \( C \) is the boundary of triangle with vertices at \((0, 0, 0), \ (1, 0, 0), \ (1, 1, 0)\).

**Solution:** Here,

\[ \mathbf{\nabla} \times \mathbf{\tilde{f}} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \\ \end{vmatrix} = 0 \mathbf{i} + j + 2(x - y) \mathbf{k} \]

\[ \therefore \oint_C \mathbf{\tilde{f}} \cdot d\mathbf{\hat{r}} = \oint_S \mathbf{\nabla} \times \mathbf{\tilde{f}} \cdot \mathbf{n} \, dS = \oint_S 2(x - y) \, dy \, dx \]

Note here the equation of OB is \( y = x \), thus for \( S \), \( x \) varies from 0 to 1 and \( y \) from 0 to \( x \).
\[
\oint_C \mathbf{f} \cdot d\mathbf{r} = \int_{x=0}^{1} \int_{y=0}^{x} 2(x-y) \, dy \, dx = \int_{x=0}^{1} 2 \left( xy - \frac{y^2}{2} \right)_{y=0}^{x} \, dx
\]

\[
= \int_{x=0}^{1} 2 \left( x^2 - \frac{x^2}{2} \right) \, dx = \int_{x=0}^{1} x^2 \, dx = \frac{1}{3}
\]

**Note:** Green’s Theorem in plane is special case of Stoke’s Theorem: If \( R \) is the region in \( xy \) plane bounded by a closed curve \( C \) then this is a special case of Stoke’s theorem. In this case \( \mathbf{n} = \mathbf{k} \) and it is called vector form of Green’s theorem in plane. Vector form of Green’s theorem can be written as

\[
\iint_R (\nabla \times \mathbf{f}) \cdot \mathbf{k} \, dR = \oint_C \mathbf{f} \cdot d\mathbf{r}
\]

**Example 65:** Evaluate \( \oint_C [(y - \sin x) \, dx + \cos x \, dy] \), where \( C \) is the triangle having vertices \((0, 0), \left(\frac{\pi}{2}, 0\right) \) and \( \left(\frac{\pi}{2}, 1\right) \) (i) directly (ii) by using Green’s theorem in plane \[\text{[KUK 2011]}\]

**Solution:**

(i) Here \( \oint_C [(y - \sin x) \, dx + \cos x \, dy] \)

\[
= \oint_C [(y - \sin x) \mathbf{i} + \cos x \mathbf{j}] \cdot (dx \mathbf{i} + dy \mathbf{j})
\]

\[
= \oint_C \mathbf{f} \cdot d\mathbf{r}
\]

where \( \mathbf{f} = [(y - \sin x) \mathbf{i} + \cos x \mathbf{j}] \) and \( d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} \) and \( C \) is triangle OAB

Now \( \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint_C [(y - \sin x) \, dx + \cos x \, dy] \)

\[
= \int_{OA} [(y - \sin x) \, dx + \cos x \, dy] + \int_{AB} [(y - \sin x) \, dx + \cos x \, dy]
\]

\[
+ \int_{BO} [(y - \sin x) \, dx + \cos x \, dy] \quad \ldots (1)
\]

On OA: \( y = 0 \), ∴ \( \int_{OA} [(y - \sin x) \, dx + \cos x \, dy] = \int_{OA} [-\sin x \, dx] = \int_{x=0}^{\pi/2} -\sin x \, dx = -1 \)

On AB: \( x = \frac{\pi}{2} \), ∴ \( \int_{AB} [(y - \sin x) \, dx + \cos x \, dy] = 0 \)

On BO: \( y = \frac{2}{\pi} x \), ∴ \( \int_{BO} [(y - \sin x) \, dx + \cos x \, dy] = \frac{2}{\pi} \int_{x=0}^{\pi/2} \left( \frac{2}{\pi} x - \sin x \right) dx + \cos x \frac{2}{\pi} \, dx \)

\[
= \left[ \frac{2}{\pi} x^2 + \cos x + \frac{2}{\pi} \sin x \right]_{x=0}^{\pi/2} = 1 - \left[ \frac{\pi}{4} + \frac{2}{\pi} \right] \]
Substituting these values in (1), we get

$$f_C \vec{r} \cdot d\vec{r} = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

(ii) By Green’s Theorem

$$f_C (f_1 \, dx + f_2 \, dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \, dxdy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (-\sin x - 1) \, dy \, dx = \int_{x=0}^{\pi/2} (-\sin x - 1) \big|_{y=0}^{2x/\pi} \, dx$$

$$= \int_{x=0}^{\pi/2} -\frac{2}{\pi} x (\sin x + 1) \, dx = -\frac{2}{\pi} \int_{x=0}^{\pi/2} (\sin x + x) \, dx$$

$$= -\frac{2}{\pi} \left[\sin x + x^2/2\right]_{x=0}^{\pi/2} = -\frac{2}{\pi} \left[\left(-0 + 1 + \frac{\pi^2}{8}\right) - (0 + 0 + 0)\right]$$

$$= -\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$

Example 66: Verify Green’s theorem in the plane for $f_C (xy + y^2) \, dx + x^2 \, dy$, where $C$ is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution:

$$f_C (xy + y^2) \, dx + x^2 \, dy = \int_{OBA} [(xy + y^2) \, dx + x^2 \, dy] + \int_{OA} [(xy + y^2) \, dx + x^2 \, dy] \quad \ldots (1)$$

Along curve OBA: $y = x^2$ $\therefore dy = 2x \, dx$

$$\therefore \int_{OBA} [(xy + y^2) \, dx + x^2 \, dy] = \int_{x=0}^{1} [(x^3 + x^4) \, dx + 2x^3 \, dx] = \frac{19}{20}$$

Along curve AO: $y = x$ $\therefore dy = dx$

$$\therefore \int_{AO} [(xy + y^2) \, dx + x^2 \, dy] = \int_{x=0}^{1} [(x^2 + x^2) \, dx + x^2 \, dx] = \int_{x=0}^{1} [3x^2] \, dx = -1$$

$$\therefore \text{from (1),} \quad f_C (xy + y^2) \, dx + x^2 \, dy = \frac{19}{20} - 1 = -\frac{1}{20} \quad \ldots (2)$$

Here $f_1 = xy + y^2$, $f_2 = x^2$

$$\Rightarrow \frac{\partial f_1}{\partial y} = x + 2y, \quad \frac{\partial f_2}{\partial x} = 2x$$

By Green’s theorem,

$$f_C (f_1 \, dx + f_2 \, dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \, dxdy$$

$$= \iint_S (2x - x - 2y) \, dxdy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} (x - 2y) \, dy \, dx = \int_{x=0}^{1} [xy - y^2]_{y=x^2}^{y=x} \, dx$$

$$= \int_{x=0}^{1} (x^4 - x^3) \, dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20} \quad \ldots (3)$$

Equation (2) and (3) verify the result.
1. Verify Green’s theorem for \(\int_C [(3x - 8y^2)dx + (4y - 6xy)dy]\) where \(C\) is the boundary of the region bounded by \(x = 0, y = 0\) and \(x + y = 1\). [KUK 2007]

2. Verify Green’s theorem in plane for \(\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]\), where \(C\) is the boundary of the region defined by \(y = \sqrt{x}\) and \(y = x^2\). [KUK 2008]

3. Apply Green’s theorem to evaluate \(\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]\), where \(C\) is the boundary of the area enclosed by \(x = 0, y = 0\) and \(x = 1\). [KUK 2010]

4. Evaluate the surface integral \(\iint_S \text{curl} \vec{F} \cdot \hat{n} \, dS\) by transforming it into a line integral, \(S\) being that part of the surface of the paraboloid \(z = (1 - x^2 - y^2)\) for which \(z \geq 0\) and \(\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}\). [KUK 2008]

5. Using Stoke’s theorem, evaluate \(\int_C [(x + y)dx + (2x - z)dy + (y + z)dz]\), where \(C\) is the boundary of the triangle with vertices \((2, 0, 0), (0, 3, 0)\) and \((0, 0, 6)\). [KUK 2009]

6. Verify Stoke’s theorem for a vector field defined by \(\vec{F} = -y^3\hat{i} + x^3\hat{j}\), in the region \(x^2 + y^2 \leq 1, z = 0\).

17.14 GAUSS’S DIVERGENCE THEOREM (Relation between Volume and Surface Integral)

**Statement:** Suppose \(V\) is the volume bounded by a closed piecewise smooth surface \(S\). Suppose \(\vec{f}(x, y, z)\) is a vector function which is continuous and has continuous first partial derivatives in \(V\). Then

\[
\iiint_V \nabla \cdot \vec{f} \, dV = \iint_S \vec{f} \cdot \hat{n} \, dS
\]

where \(\hat{n}\) is the outward unit normal to the surface \(S\).

**In other words:** The surface integral of the normal component of a vector \(\vec{f}\) taken over a closed surface is equal to the integral of the divergence of \(\vec{f}\) over the volume enclosed by the surface.

**Divergence Theorem in Cartesian Coordinates**

If \(\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\) then \(\text{div} \vec{f} = \nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\). Suppose \(\alpha, \beta, \gamma\) are the angles made by the outward drawn unit normal with the positive direction of axes, then

\[
\hat{n} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k}
\]

Now, \(\vec{f} \cdot \hat{n} = f_1 \cos \alpha + f_2 \cos \beta + f_3 \cos \gamma\)

Then divergence theorem is

\[
\iiint_V \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) \, dx \, dy \, dz = \iint_S (f_1 \cos \alpha + f_2 \cos \beta + f_3 \cos \gamma) \, dS
\]

\[
= \iint_S (f_1 dydz + f_2 dzdx + f_3 dxdy)
\]

[\because \cos \alpha \, dS = dydz \text{ etc.}]

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**Proof:** Let \( \mathbf{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k} \) where \( f_1, f_2 \) and \( f_3 \) and their derivatives in any direction are finite and continuous.

Suppose \( S \) is a closed surface such that it is possible to choose rectangular cartesian co-ordinate system such that any line drawn parallel to coordinate axes does not cut \( S \) in more than two points.

Let \( R \) be the orthogonal projection of \( S \) on the \( xy \)-plane. Any line parallel to \( z \)-axis through a point of \( R \) meets the boundary of \( S \) in two points. Let \( S_1 \) and \( S_2 \) be the lower and upper portions of \( S \). Let the equations of these portions be

\[
z = \Phi_1(x, y) \quad \text{and} \quad z = \Phi_2(x, y)
\]

where \( \Phi_1(x, y) \geq \Phi_2(x, y) \)

Consider the volume integral

\[
\iiint_V \frac{\partial f_3}{\partial z} \, dV = \iiint_V \frac{\partial f_3}{\partial z} \, dx \, dy \, dz = \iiint_{V_{z=\Phi_1(x,y)}} \left( \frac{\partial f_3}{\partial z} \right) \, dx \, dy
\]

\[
\iiint_V \frac{\partial f_3}{\partial z} \, dV = \iint_{\phi_2(x,y)} [f_3(x,y,z)] \, dx \, dy = \iint (f_3(x,y,\Phi_1) - f_3(x,y,\Phi_2)) \, dx \, dy \quad \ldots (1)
\]

Let \( \mathbf{n}_1 \) be the unit outward drawn vector making an acute angle \( \gamma_1 \) with \( \mathbf{k} \) for the upper position \( S_1 \) as shown in the figure.

Now projection \( dx \, dy \) of \( dS_1 \) on the \( xy \) plane is given as \( dx \, dy = dS_1 \cos \gamma = dS_1 \mathbf{k} \cdot \mathbf{n}_1 = \mathbf{k} \cdot \mathbf{n}_1 \, dS_1 \)

Now \( \iint_S f_3(x,y,\Phi_1) \, dx \, dy = \iint_{S_1} f_3 \mathbf{k} \cdot \mathbf{n}_1 \, dS_1 \quad \ldots (2) \)

Similarly if \( \mathbf{n}_2 \) be the unit outward drawn normal to the lower surface \( S_2 \) making an angle \( \gamma_2 \) with \( \mathbf{k} \). Obviously \( \gamma_2 \) is an obtuse angle

\[
\therefore \quad dx \, dy = dS_2 \cos(\pi - \gamma_2) = -dS_2 \cos \gamma_2 = -\mathbf{k} \cdot \mathbf{n}_2 \, dS_2
\]

\[
\therefore \quad \iint_S f_3(x,y,\Phi_2) \, dx \, dy = -\iint_{S_2} f_3 \mathbf{k} \cdot \mathbf{n}_2 \, dS_2 \quad \ldots (3)
\]

From (1), (2) and (3), we have

\[
\iiint_V \frac{\partial f_3}{\partial z} \, dV = \iint_{S_1} f_3 \mathbf{k} \cdot \mathbf{n}_1 \, dS_1 + \iint_{S_2} f_3 \mathbf{k} \cdot \mathbf{n}_2 \, dS_2 = \iint_S f_3 \mathbf{k} \cdot \mathbf{n} \, dS \quad \ldots (4)
\]

Similarly by projecting \( S \) on the other coordinate planes

\[
\iiint_V \frac{\partial f_2}{\partial y} \, dV = \iint_S f_2 \mathbf{j} \cdot \mathbf{n} \, dS \quad \ldots (5)
\]

And \( \iiint_V \frac{\partial f_1}{\partial x} \, dV = \iint_S f_1 \mathbf{i} \cdot \mathbf{n} \, dS \quad \ldots (6) \)
Adding (4), (5) and (6), we get

\[ \iiint_V \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) \, dV = \iint_S \left( f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \right) \cdot \hat{n} \, dS = \iint_S \vec{f} \cdot \hat{n} \, dS \]

**Note:** With the help of this theorem we can express volume integral as surface integral or vice versa.

### 17.15 GREEN’S THEOREM (For Harmonic Functions)

**Statement:** If \( \Phi \) and \( \psi \) are two scalar point functions having continuous second order derivatives in a region \( V \) bounded by a closed surface \( S \), then

\[ \iiint_V \left( \Phi \nabla^2 \psi - \psi \nabla^2 \Phi \right) \, dV = \iint_S \left( \Phi \nabla \psi - \psi \nabla \Phi \right) \cdot \hat{n} \, dS \]

**Proof:** By Gauss’s divergence theorem,

\[ \iiint_V \nabla \vec{f} \, dV = \iint_S \vec{f} \cdot \hat{n} \, dS \quad \ldots (1) \]

Take \( \vec{f} = \Phi \nabla \psi \) so that \( \nabla \vec{f} = \nabla (\Phi \nabla \psi) \)

\[ = \Phi (\nabla \nabla \psi) + \nabla \Phi \cdot \nabla \psi \]

\[ = \Phi \nabla^2 \psi - \nabla \Phi \cdot \nabla \psi \quad \ldots (2) \]

Now \( \vec{f} \cdot \hat{n} = (\Phi \nabla \psi) \cdot \hat{n} \). Using this and (2) in (1), we get

\[ \iiint_V \left[ \Phi \nabla^2 \psi - \nabla \Phi \cdot \nabla \psi \right] dV = \iint_S \left( \Phi \nabla \psi \right) \cdot \hat{n} \, dS \quad \ldots (3) \]

Again starting as above by interchanging \( \Phi \) and \( \psi \), we obtain as in (3)

\[ \iiint_V \left[ \psi \nabla^2 \Phi - \nabla \psi \cdot \nabla \Phi \right] dV = \iint_S \left( \psi \nabla \Phi \right) \cdot \hat{n} \, dS \quad \ldots (4) \]

Subtracting (4) from (3), we get

\[ \iiint_V \left( \Phi \nabla^2 \psi - \psi \nabla^2 \Phi \right) \, dV = \iint_S \left( \Phi \nabla \psi - \psi \nabla \Phi \right) \cdot \hat{n} \, dS \quad \ldots (5) \]

**Another Form of Green’s Theorem:**

\( \frac{\partial \Phi}{\partial n} \) and \( \frac{\partial \psi}{\partial n} \) denote the direction of derivative of \( \Phi \) and \( \psi \) along the outward drawn normal at any point of \( S \).

\[ \nabla \Phi = \frac{\partial \Phi}{\partial n} \hat{n} \] and \[ \nabla \psi = \frac{\partial \psi}{\partial n} \hat{n} \]

\[ \therefore \quad \Phi \nabla \psi - \psi \nabla \Phi = \left( \Phi \frac{\partial \psi}{\partial n} \right) \hat{n} - \left( \psi \frac{\partial \Phi}{\partial n} \right) \hat{n} \]

or \[ (\Phi \nabla \psi - \psi \nabla \Phi) \cdot \hat{n} = \left( \Phi \frac{\partial \psi}{\partial n} \hat{n} - \psi \frac{\partial \Phi}{\partial n} \hat{n} \right) \cdot \hat{n} = \Phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \Phi}{\partial n} \]

\[ \therefore \quad \text{Equation (5) becomes} \]

\[ \iiint_V \left( \Phi \nabla^2 \psi - \psi \nabla^2 \Phi \right) \, dV = \iint_S \left( \Phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \Phi}{\partial n} \right) \, dS \]
Example 67: Evaluate \( \int_S \vec{f} \cdot \hat{n} \, dS \) where \( \vec{f} = 4xy \hat{i} + yz \hat{j} - zx \hat{k} \) and \( S \) is the surface of the cube bounded by the planes \( x = 0, x = 2, y = 0, y = 2, z = 0, z = 2 \).

Solution: Here \( \vec{f} = 4xy \hat{i} + yz \hat{j} - zx \hat{k} \). By Gauss’s divergence theorem

\[
\int_S \vec{f} \cdot \hat{n} \, dS = \iiint_V \nabla \vec{f} \, dV
\]

\[
= \iiint_V \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) (4xy \hat{i} + yz \hat{j} - zx \hat{k}) \, dV
\]

\[
= \iiint_V (4y + z - x) \, dV = \int_{x=0}^{z=2} \int_{y=0}^{z=2} \int_{z=0}^{x=2} (4y + z - x) \, dy \, dx \, dz
\]

\[
= \int_{x=0}^{z=2} \left[ \int_{y=0}^{z=2} (4y + z - x) \, dy \right] \, dx \, dz
\]

\[
= \int_{x=0}^{z=2} (4y^2 + 2y - 2xy) \, dx \, dy \, dz
\]

\[
= \int_{x=0}^{z=2} (20 - 4x) \, dx \, dy \, dz
\]

\[
= \left[ 20x - 2x^2 \right]_{x=0}^{x=2} = 32
\]

Example 68: Use Gauss theorem to show that \( \int_S [(x^3 - yz) \hat{i} - 2x^2y \hat{j} + 2 \hat{k}] \cdot \hat{n} \, dS = \frac{a^5}{3} \)

where \( S \) denotes the surface of the cube bounded by the planes, \( x = 0, x = a, y = 0, y = a, z = 0, z = a \).

Solution: By Gauss’s divergence theorem

\[
\int_S [(x^3 - yz) \hat{i} - 2x^2y \hat{j} + 2 \hat{k}] \cdot \hat{n} \, dS = \iiint_V \nabla \cdot (x^3 - yz) \hat{i} - 2x^2y \hat{j} + 2 \hat{k} \, dV
\]

\[
= \int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} (3x^2 - 2x^2) \, dx \, dy \, dz
\]

\[
= \int_{x=0}^{a} \int_{y=0}^{a} \left[ x^2 \right]^a_0 \, dy \, dz
\]

\[
= \int_{x=0}^{a} \int_{y=0}^{a} \left( \frac{a^3}{3} \right) \, dy \, dz
\]

\[
= \int_{x=0}^{a} \int_{y=0}^{a} \left( \frac{a^3}{3} \right) \, dy \, dz
\]

\[
= \int_{x=0}^{a} \left( \frac{a^3}{3} \right) \, dy \, dz
\]

\[
= \int_{x=0}^{a} \left( \frac{a^3}{3} \right) \, dy \, dz
\]

\[
= \int_{x=0}^{a} \left( \frac{a^3}{3} \right) \, dy \, dz
\]

\[
= \left[ \frac{a^3}{3}y \right]_{y=0}^{y=a} = \frac{a^5}{3}
\]

Example 69: Evaluate \( \int_S \vec{f} \cdot \hat{n} \, dS \) with the help of Gauss theorem for \( \vec{f} = 6z \hat{i} + (2x + y) \hat{j} - x \hat{k} \) taken over the region \( S \) bounded by the surface of the cylinder \( x^2 + z^2 = 9 \) included between \( x = 0, y = 0, z = 0 \) and \( y = 8 \).

Solution: \( \vec{f} = 6z \hat{i} + (2x + y) \hat{j} - x \hat{k} \)

\[
\nabla \cdot \vec{f} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) (6z \hat{i} + (2x + y) \hat{j} - x \hat{k}) = 1
\]

By Gauss’s divergence theorem,

\[
\int_S \vec{f} \cdot \hat{n} \, dS = \iiint_V 1 \, dx \, dy \, dz
\]
\[ \int_{x=0}^{3} \int_{y=0}^{8} \int_{z=0}^{9-x^2} dz \, dy \, dx = \int_{x=0}^{3} \int_{y=0}^{8} [z]_{z=0}^{9-x^2} dy \, dx \]

\[ = \int_{x=0}^{3} \int_{y=0}^{8} \sqrt{9-x^2} \, dy \, dx \]

\[ = \int_{x=0}^{3} [\sqrt{9-x^2} \cdot y]_{y=0}^{8} dx = \int_{x=0}^{3} 8 \sqrt{9-x^2} \, dx \]

\[ = 8 \left( \frac{x \sqrt{9-x^2}}{2} + \frac{9}{2} \sin^{-1} \frac{x}{3} \right)_{x=0}^{3} = 18 \pi \]

Example 70: Evaluate \( \iint_S (xydz + ydzdx + zdxdy) \) where \( S \) is the surface of the sphere \( x^2 + y^2 + z^2 = a^2 \). [KUK 2011, 2009]

Solution: By Gauss’s divergence theorem

\[ \iint_S (xydz + ydzdx + zdxdy) = \iiint_V \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dx \, dy \, dz = \iiint_V 3 \, dx \, dy \, dz \]

\[ = 3 \iiiint_V dx \, dy \, dz = 3 \times \text{volume of the sphere } x^2 + y^2 + z^2 = a^2 \]

\[ = 3 \times \frac{4}{3} \pi a^3 = 4 \pi a^3 \]

Example 71: Show that \( \oint_S \hat{n} \, dS = 0 \) for any closed surface \( S \).

Solution: Let \( C \) be any closed vector.

\[ \therefore \quad C \oint_S \hat{n} \, dS = \oint_S \hat{n} \, dS = \iiint_V \text{div} \, C \, dV \]

\[ \therefore \quad C \oint_S \hat{n} \, dS = 0 \quad [\because \quad C \text{ is constant, therefore } \text{div} \, C = 0] \]

\[ \Rightarrow \quad \oint_S \hat{n} \, dS = 0 \]

Example 72: Prove that \( \iiint_V \frac{1}{r^2} \, dV = \iiint_S \frac{\hat{r} \cdot \hat{n}}{r^2} \, dS \), where \( \hat{r} = x \hat{i} + y \hat{j} + z \hat{k} \) and \( |\hat{r}| = r \).

Solution: \( \iiint_S \frac{\hat{r}}{r^2} \, dS = \iiint_S \frac{\hat{r}}{r^2} \cdot \hat{n} \, dS = \iiint_V \left( \nabla \cdot \frac{\hat{r}}{r^2} \right) dV \)

\[ \text{(1)} \]

Now,

\[ \nabla \left( \frac{1}{r^2} \right) = \frac{1}{r^2} \left( \nabla \cdot \hat{r} \right) + \hat{r} \cdot \nabla \left( \frac{1}{r^2} \right) \]

\[ \text{(2)} \]

Also,

\[ \hat{r} = x \hat{i} + y \hat{j} + z \hat{k} \quad \Rightarrow \quad r^2 = x^2 + y^2 + z^2 \]

\[ \therefore \quad 2r \frac{\partial r}{\partial x} = 2x; \quad 2r \frac{\partial r}{\partial y} = 2y; \quad 2r \frac{\partial r}{\partial z} = 2z \]

\[ \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{(3)} \]

Now,

\[ \nabla \left( \frac{1}{r^2} \right) = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left( \frac{1}{r^2} \right) = -2 \left( \frac{1}{r^3} \left( \hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z} \right) \right) \]

\[ \text{[using (3)]} \]

\[ = \frac{-2}{r^4} \left( x \hat{i} + y \hat{j} + z \hat{k} \right) = \frac{-2}{r^4} \hat{r} \]
Also, \( \nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3 \)

Substituting these values in (2), we have
\[
\nabla \cdot \left( \frac{\mathbf{r}}{r^2} \right) = \frac{1}{r^2} \mathbf{\hat{r}} \cdot \mathbf{3} + \mathbf{\hat{r}} \cdot \left( -\frac{2}{r^4} \right) \mathbf{\hat{r}} = \frac{3}{r^2} - \frac{2}{r^4} r^2 = \frac{1}{r^2} \quad [\nabla \cdot \mathbf{\hat{r}} = r^2]
\]
\( \therefore \) From (1), \( \iiint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^2} \, dS = \iiint_V \frac{1}{r^2} \, dV \)

**ASSIGNMENT 10**

1. Find \( \iiint_S \mathbf{F} \cdot \mathbf{n} \, dS \) where \( \mathbf{F} = (2x + z) \mathbf{\hat{i}} - (xz + y) \mathbf{\hat{j}} + (y^2 + 2z) \mathbf{\hat{k}} \) and S is the surface of the sphere having centre \((3, -1, 2)\) and radius 3 units. [KUK 2006]

2. Use Divergence theorem to evaluate \( \iiint_S \mathbf{F} \cdot dS \) where \( \mathbf{F} = x^3 \mathbf{\hat{i}} + y^3 \mathbf{\hat{j}} + z^3 \mathbf{\hat{k}} \) and S is the outer surface of the sphere \( x^2 + y^2 + z^2 = 1 \). [KUK 2007]

3. Verify Divergence theorem for \( \mathbf{F} = (x^2 - yz) \mathbf{\hat{i}} + (y^2 - zx) \mathbf{\hat{j}} + (z^2 - xy) \mathbf{\hat{k}} \) taken over rectangular parallelopiped \( 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c \). [KUK 2010]

**ANSWERS**

**ASSIGNMENT 1**

1. \(-4(\mathbf{i} + 2\mathbf{j})\) 3. \((x - a/\sqrt{2}) = (y - a/\sqrt{2}) = (z - a/4 \tan \alpha)/\sqrt{2} \tan \alpha\)

4. \([t \mathbf{i} + 2\mathbf{j} - (2t - 3)\mathbf{k}] / \sqrt{\left(5t^2 - 12t + 13\right)} ; \frac{1}{2} (2 \mathbf{i} + 2\mathbf{j} + \mathbf{k})\)

5. (a) \(u \alpha^2 \sec \alpha\) (b) \(a^3 \tan \alpha\); \((- \cos \alpha \sin t) \mathbf{i} + \cos \alpha \cos t) \mathbf{j} + \sin \alpha \mathbf{k}\)

6. (a) \(p \mathbf{i} + (p + 2q) \mathbf{j} + (p + q) \mathbf{k}\); \(\frac{1}{\sqrt{5}} (-\mathbf{i} - \mathbf{j} + 2 \mathbf{k})\) (b) \((p + q) \mathbf{i} + q \mathbf{j} + 2q \mathbf{k}; \frac{1}{\sqrt{5}} (2\mathbf{j} - \mathbf{k})\)

17. (a) \(ab / (a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}\) (b) \(1/4\sqrt{2}\)

**ASSIGNMENT 2**

1. \(v = \sqrt{37}\) and \(a = 325\) at \(t = 0\) 3. \(\frac{8}{7} \sqrt{14}\); \(\frac{1}{7} \sqrt{14}\) 4. \(a = \pm \frac{1}{\sqrt{6}}\) 7. \(\frac{70}{\sqrt{29}}\); \(\sqrt{\frac{436}{29}}\)

8. 21.29 knots/hr. in the direction \(74^0 \ 47'\) South of East 9. \(\sqrt{17}\) meter per hour in the direction \(\tan^{-1}(0.25)\) North of East

**ASSIGNMENT 3**

1. (a) \(2(x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) / (x^2 + y^2 + z^2)\) 3. \(\frac{15}{\sqrt{17}}\) 4. \(\frac{37}{3}\) 5. \(\frac{1}{3} (2 \mathbf{i} + 2 \mathbf{j} - \mathbf{k})\) 6. 11

7. \(-\frac{1}{\sqrt{30}}\) 8. \(-1/\sqrt{22}\) 9. \(-\frac{8}{\sqrt{21}}\) 10. \(\lambda = 4\) and \(\mu = 1\)
ASSIGNMENT 4

2. (a) 80 (b) \( e^{xyz} (x(z - y)i + y(x - z)j + z(y - x)k) \)

3. \( a = -2; \ 4x(z - xy)i + y(1 - 2z + 4xy)j + (2x^2 + y^2 - z^2 - z)k \)

9. (a) \( \frac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}}; \quad n = \frac{1}{2} \)

11. (i) \( 2(y^3 + 3x^2y - 6xy^2)z \ i + 2(3xy^2 + x^3 - 6x^2y)z \ j + 2(xy^2 + x^3 - 3x^2y)z \ k \)  
   (ii) Zero

13. (i) 0 (ii) \( 2(x + z) \ j + 2y \ k \)

ASSIGNMENT 5

1. \( \frac{226}{3} \ i + 360 \ j - 42 \ k \)

2. \( -2 \ i + 3 \ j - 3 \ k \)

4. \( \ddot{v} = 6 \sin 2t \ i + 4(\cos 2t - 1) \ j + 8t^2 \ k \) and \( \ddot{r} = 3(1 - \cos 2t) \ i + 2 \sin 2t \ j + \frac{8t^3}{3} \ k \)

ASSIGNMENT 6

1. \( \frac{8}{35} \)  
2. 16, 16  
4. 35  
5. \( \frac{\pi^{3/2}}{3} \)

6. \( \left(2 - \frac{\pi}{4}\right)i - \left(\pi - \frac{1}{2}\right)j \)

ASSIGNMENT 7

2. 81  
3. 3/2  
5. 8

ASSIGNMENT 8

1. 128  
2. \( 128i - 24j + 384k \)

ASSIGNMENT 9

3. 4/3  
5. 21

ASSIGNMENT 10

1. 108\pi  
2. \( 56\pi a^2/9 \)